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SOME APPLICATIONS OF GROUPS TO GEOMETRY[1]

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Introduction

The subject of groups is of wide appeal, both for its intrinsic interest and the range and importance of its applications—applications to geometry, to algebra,—notably in the theory of equations and the invariant theory,—to mathematical physics, to differential equations, to crystal structure, to the principles of design. Yet it is worthy of remark that one can start from elementary notions and reach the frontier in a comparatively short time without the aid of other branches of analysis. And the postulates are few—Dickson's set consists of four, while for finite groups of collineations, a single postulate suffices.

In the brief time allotted to this paper, it is possible to sketch but one phase of the applications even to geometry. I only mention in passing the great field of the Lie theory where continuous groups are so effectively brought into play. Again the Galois theory of equations has its geometric counterpart in such problems as those associated with the inflexions of the cubic curve, the bitangents of the plane quartic and the twenty-seven lines on the cubic surface. There is the large and increasing domain of Cremona geometry and Cremona groups. Klein has made the theory of groups of linear transformations the basis of the classification of the various geometries which occur as special cases of general projective geometry. Indeed the theory of linear transformations is the very soul of projective geometry and its algebraic dual, the classical invariant theory. In spite of this amazing richness of application, many scholars are content to explore the abstract theory of the groups themselves, indifferent to the applications or perhaps serene in the delusion that some of their results are immune to application.

I cannot refrain from giving voice here to my own and other geometers' keen disappointment that in the recent invaluable *Report on Algebraic Geometry*,[2] space could not have been found for the important subject of collineation groups, more especially since the keynote of the Report is transformation.

§1. *The Problem of Symmetry*

I propose to consider briefly the application of collineation groups, binary and ternary, to the geometry of plane curves, beginning with the metrical question of symmetry. We recognize in the plane two elementary types of symme-

[1] Read at the summer meeting of the Mathematical Association of America at Boulder, Colorado, August 27, 1929.

[2] National Research Council Bulletin No. 63.

try (*a*) with respect to a line (*b*) with respect to a point. These are illustrated respectively by the ordinary parabola $y^2 = x$, and the cubical parabola $y = x^3$. In the latter case, the center of symmetry is defined to be the center of the curve. Their utility for tracing the curves is obvious, for in either case the points of the curve are paired in the symmetry—paired in a mutual way so that either point of a pair determines the other.

It is desirable now to have a mechanism that will enable us, given one point of a pair, to construct its partner. Metrically stated, we are given one end and the middle point of a line segment and are required to construct the other end. But in projective language the middle point of a segment and the infinite point of the segment (produced) divide the end-points harmonically and the problem becomes: Given three points of a line, to construct the harmonic conjugate of one with respect to the other two. The projective view of this operation, which is called *reflexion*, is illustrated by a conic and a polar point and line which are not incident: Any line through the pole cuts the conic in a pair of points which are harmonically separated from the pole by the point in which the line cuts the polar line. We say that the conic is reflected into itself, calling the pole the center and the polar line the axis of the reflexion.

It appears now that from the projective point of view the two elementary types of symmetry are identical: In the first the axis of symmetry is the axis of reflexion while the center of reflexion is a point at infinity in the direction of a perpendicular to the axis; in the other the axis of reflexion is the line at infinity while the center of reflexion coincides with the center of symmetry. Thus we may summarize as follows:

If a curve is symmetrical with respect to a line, it is invariant under a reflexion of which the line is the axis. And if a curve is symmetrical with respect to a point, it is invariant under a reflexion of which the point is the center. Conversely, if a curve is invariant under a reflexion, it may be projected indifferently into one having (*a*) *an axis of symmetry or* (*b*) *a center of symmetry.*

The geometrical apparatus for constructing a pair of points interchanged by a reflexion is the complete quadrangle in the general projective case; in the metrical case when the points are symmetrical with respect to a line or a point, the familiar Euclidean constructions are available. As an algebraic means of determining such a pair of points, we resort to a *transformation*. Thus in the metrical case of the parabola, if the reflexion carries the point (x, y) into the point (x', y'), the relation between the coordinates of the two points is

$$R_1: \quad x' = x, \quad y' = -y.$$

These are the equations of a reflexion which we regard as the operation which carries the point (x, y) into the point (x', y'). A second application of the reflexion restores the point to its original position. Thus a reflexion is of period two and in fact any linear transformation of period two in the plane is a reflexion. Under a reflexion the center and all points of the axis and hence the axis itself and all lines through the center are fixed, while all other points and lines of the

plane are paired. Consider now a second reflexion, in the y-axis, whose defining equations are

$$R_2: \quad x' = -x, \quad y' = y.$$

If R_1 carries a point P into a point P' and R_2 carries P' into P'', then P'' can be obtained from P by means of the transformation

$$R_3: \quad x' = -x, \quad y' = -y.$$

R_3, which is the resultant or *product* of R_1 and R_2, is obviously again a reflexion —in the origin. Or we may regard R_3 as a rotation about the origin through the angle 180°. Finally we associate with the reflexions the identical transformation, or simply the *identity*

$$1: \quad x' = x, \quad y' = y,$$

which leaves every point fixed. The combined effect of these reflexions when applied to an arbitrary point of the plane is to carry the point into four positions, including the original. Such a set of linear transformations, finite in number, all possible combinations of which carry a point into different positions, ultimately restoring it to the original one, are said to form a collineation group. Or, somewhat more conventionally,

A set of collineations, finite in number, the product of any two of which produces the same effect as some one collineation of the set, constitute a finite collineation group. It is understood that the several powers of each collineation (successive products of the collineation by itself) are included in the definition. Thus in the group above, we have the relations

$$R_i^2 = 1, R_iR_j = R_k, \qquad i,j,k, = 1,2,3, i \neq j \neq k.$$

The individual collineations are called *elements* and the *order* of a group is the number of elements it contains. A group of order n we shall uniformly denote by G_n. Two points which can be transformed into each other by the elements of the group are said to be *conjugate*. In the sequel "group" shall be understood to mean finite collineation group.

As immediate consequences of the definition we have

1°. *Every collineation S_i of a finite group is periodic, i.e. for some integer p, $S_i^p = 1$*, otherwise the order of the group would be infinite. If p is the least integer for which this relation holds, S_i is of period p.

2°. *The identity is included in every group*—a corollary of 1°.

3°. *The inverse S_i^{-1} of every element S_i belongs to the group.* For if p be the period of S_i, $S_i^{p-1} = S_i^{-1}$.

We return now to the question of symmetry. Suppose a curve has two axes of symmetry, inclined at an angle θ. The curve is then invariant under two reflexions, say T and T'. The product of the two is obviously a rotation, say S, about the origin through an angle 2θ. Hence the curve is also invariant under S, which simply means that under repeated applications of S a point of the curve always goes again into a point of the curve. If now $2n\theta = 360°$, S repeated will

carry an arbitrary point of the curve into n positions, including the original. In other words $S, S^2, \cdots, S^n \equiv 1$, will constitute a group of order n which transforms the curve into itself. S and T in combination generate a group of order $2n$, the ternary dihedral G_{2n}, which leaves the curve invariant. The group contains n reflexions whose axes are equispaced about the origin, i.e. the curve has n equispaced axes of symmetry. Evidently the dihedral group, in the metrical setting here described, leaves unaltered each circle of the system having the origin for common center. An arbitrary point of any invariant curve is one of a set of $2n$ conjugate points cut from the curve by a circle of this concentric system.

In the metrical representation of the ternary dihedral group now employed, the centers of the n reflexions lie on the line at infinity, which is a fixed line as the origin is a fixed point. In a projective representation, the dihedral group has likewise a fixed line, the line of centers, and a fixed point, the intersection of the n axes of reflexion. It also leaves invariant a pencil of double-contact conics, with respect to every (proper) member of which the fixed point and the line are pole and polar.

It is plain now that there is an intimate connection between multiple symmetry and dihedral groups, for a dihedral group can always be given the metrical representation described above. Thus

If a curve is invariant under a dihedral G_{2n}, it may be projected into one having n axes of symmetry, equispaced about a point. Conversely, if a curve is symmetrical with respect to n equispaced axes, it admits a dihedral G_{2n}.

Again, *a curve of order n, a conic alone excepted, may not admit a dihedral group of order greater than $2n$, i.e the maximum number of lines with respect to which an algebraic curve may be symmetrical is $2n$. When n is even however, the intersection of two or more axes of symmetry is a center of the curve.* For if the curve is invariant under a dihedral G_{2n}, any general set of conjugate points on the curve is cut out by one of the invariant conics (circles) which cuts the curve in a maximum of $2n$ points.

There is however at least one algebraic curve for each value of n possessing this maximum symmetry. Indeed there is at least one rational curve—one cubic (the general rational cubic) one quartic (the projective lemniscate) two quintics, one sextic and three septimics. The general formula for rational curves is[1]

$$x_1 = t^r \qquad r + s = n,$$
$$x_2 = t^s \qquad 1 \geq r \neq s,$$
$$x_3 = t^n + 1,$$

where r, s and n are integers and have no common factor.

We may now state a further theorem about multiple symmetry:

[1] Winger, American Journal of Mathematics vol. 36 (1914), p. 66. To plot the curves so that the symmetry appears, absolute or circular coordinates x, $\bar{x}$ must be used, writing $x = x_1/x_3$, $\bar{x} = x_2/x_3$, while to the parameter t are to be assigned complex numbers of absolute value 1.

If an algebraic curve have two axes of symmetry, it is invariant under a dihedral group. For the curve is then invariant under two reflexions, whose product is a rotation. This rotation must be periodic, else the curve would have an infinite number of axes of symmetry, whereas the maximum number is n. If the period of the rotation is m, then the two reflexions generate a dihedral group of order[1] $2m$ and the curve has m axes of symmetry.

A curve may admit a rotation into itself of period n without admitting a reflexion. Such a curve possesses a sort of "one-sided" symmetry which is admirably illustrated by a swastika or a cross section of an anemometer. For example the quartic curve

$$xy^3 - x^3y = 1$$

is invariant under the G_4 of rotations generated by $x' = -y$, $y' = x$.

A very interesting analysis of symmetry in design from the group standpoint, with references, is given by Speiser in his admirable book *Theorie der Gruppen von Endlicher Ordnung*, Springer, 1927.[2] Weyl in his *Gruppentheorie und Quantenmechanik* goes so far as to say that a substantial part of group theory is implicit in the art of ornamental design, which was brought to a high state of perfection by the Egyptians.

§2. *Applications to Configurations*

Since every element of a finite group is periodic, a useful theorem is the following:

Any periodic collineation in a space of n dimensions may, by a suitable choice of the reference frame, be reduced to the canonical form

$$\rho x_i' = a_i x_i, \quad i = 0, 1, \cdots, n, \tag{1}$$

where the x's are homogeneous projective coordinates and the a's (called *multipliers*) are roots of unity. If the collineation is of period n, one of the a's must be a primitive nth root. The theorem does not imply of course that all the collineations of a group can be thrown simultaneously into the form (1). When that is possible, the group is *monomial*. The distinguishing property of a monomial group is that it leaves invariant an $(n+1)$-point which is the reference frame when the group is in the canonical form. This gives a broad classification of collineation groups into two classes, those which are monomial and those which are not. Because of their comparative simplicity monomial groups have been almost ignored, but they are deserving of a better fate because of the interesting geometry associated with them.

The theorem restricts the types of collineations that may occur in a finite group to two main classes represented by (1) when (a) the multipliers are distinct and (b) two or more multipliers are the same. Thus of the five projectively distinct types of collineations in the plane only two may belong to a finite group while of the thirteen possible varieties in space but four may occur in a finite

[1] Miller, Blichfeldt, Dickson, *Finite Groups*, p. 61.

[2] See also the delightful *Lectures on the Principle of Symmetry* by Jaeger, Amsterdam, 1920.

group. Geometrically, type (a) is characterized as having a finite number of fixed points, while collineations of type (b) leave fixed all of the points of one or more linear spaces.[1]

Conjugate points and lines. In the absence of explicit statement to the contrary, we shall henceforth confine ourselves to the plane. The effect of operating on a point P with all the collineations of a group G_n, individually and collectively in any combination, is to carry P into a maximum of n positions, including the original. For each collineation either leaves P fixed or transforms it into a second point and by definition the effect of two elements operating in succession is the same as that of some one element. The set of points thus associated with P forms a set conjugate under the group. Paralleling the idea of conjugate points is the notion of conjugate elements in the group itself. Two conjugate collineations are projectively equivalent, or as we say abstractly identical, i.e. by a proper choice of the reference scheme each may be reduced to exactly the same form as the other. In symbols, if S and T are any two elements of a group and if $T^{-1}ST=S'$, where the multiplication is understood to be from left to right, S and S' are conjugate elements.

A cardinal principle for the geometry is the following: *Any two points in a set of conjugate points have exactly the same (projective) properties.* Thus if a curve is invariant under a group, a point of inflexion can go only into a point of inflexion, a double point into a double point, an undulation into an undulation, etc. If the curve has but one point of a species, that point must be fixed under the group. The principle applies also to the relation of a point to the group. Thus if a point is fixed under a collineation S, it can be carried by the group only into a fixed point of an element conjugate to S. The principle is illustrated by the relation of congruent figures in Euclidean geometry, where we think of congruent figures as having identical properties. Indeed congruent figures are those which can be transformed into each other by the group of motions—consisting of translations and rotations—that characterize Euclidean geometry.

The principle affords a means of classifying precisely the points of a curve. We are accustomed to speak somewhat loosely of a "generic point" of a curve, a vague expression used to distinguish it from points possessing some striking peculiarity such as the "singular points." A generic point on a plane cubic would signify any point except an inflexion. But the case is not so simple as that. There are 27 points at each of which a conic with 6-point contact can be drawn (sextactic points), 72 which are vertices of the 24 Poncelet triangles, a limited number which are vertices of Poncelet polygons of higher order, etc. All such points are in a strict sense special since other points of the curve do not share all of their properties. I submit the following criterion for the equivalence of points on an algebraic curve: *If the curve is not invariant under a collineation group, no two points of the curve are equivalent in a projective sense; while if the curve is invariant under a collineation group, only conjugate points may be regarded as equivalent.*

[1] All of the elements of a binary group, except the identity, are of type (a).

In the present instance the nine flexes are equivalent since they constitute a set conjugate under the G_{18} which leaves the curve invariant. The 27 sextactic points however belong in three conjugate sets of 9 points each, while the 72 vertices of the Poncelet triangles divide into four sets each containing 18 conjugate points.

Again the binomial rational curves

$$x = t^n, \quad y = t^{n-r}, \quad z = 1, \quad \text{or} \quad y^n - x^{n-r}z^r = 0$$

are invariant under infinite groups, binary and ternary, generated respectively by

$$t = at' \quad \text{and} \quad x' = a^n x, \quad y' = a^{n-r}y, \quad z' = z.$$

Each of the curves has dual singularities at $t=0, \infty$ which absorb all of the double points, flexes and double lines. Each of these points is fixed under both groups, but all other points are equivalent to each other.[1]

Special sets. Certain points of the plane may assume fewer than n positions under the operations of a G_n. Such points belong to special sets of conjugate points. Each point of a special set must be fixed under one or more collineations, hence to discover all special sets we need only examine the fixed points of the constituent elements. Suppose that a point is fixed under k (and only k) elements of G_n. Then every point conjugate to it will be fixed under k elements and the aggregate of such points will form a special set. Every point in the special set thus arises from the coincidence of k points of a general set. It follows that

The number of points in a special set is a factor of n.

Now all the elements of G_n that leave a point fixed obviously form a subgroup, since if any two elements leave a point fixed so also will their product. Hence *each point of a special set of n/k conjugate points is invariant under a subgroup G_k, of order k.* Moreover, *the set of subgroups of order k, each of which comprises all the elements that leave one point of a special set of n/k points unaltered are conjugate subgroups.* The number of conjugate subgroups in such a set may be equal to n/k or it may be less, since the subgroups need not all be distinct. The number, which in the extreme case may reduce to one, is however always a factor of n/k.

As its name indicates a collineation carries lines into lines as well as points into points. All that we have said about conjugate sets of points applies equally to conjugate sets of lines for

The configuration of fixed points and lines under the collineation (1) *is self-dual. Hence the configuration determined by every special set of conjugate lines is dual to that determined by some special set of conjugate points.*

To prove the theorem for the plane, we observe that the collineation (1), $i=1, 2, 3$, transforms the line

[1] There is an exception in the case of the conic which has no singular points. And since 0 and ∞ may be assigned as parameters to any two points of the curve, all points on a conic are equivalent according to the criterion.

$$u_1x_1 + u_2x_2 + u_3x_3 = 0$$

into

$$\rho(u_1x_1'/a_1 + u_2x_2'/a_2 + u_3x_3'/a_3) \equiv u_1'x_1' + u_2'x_2' + u_3'x_3' = 0.$$

Thus the connection between the old and new u's is

$$u_i' = \rho u_i/a_i, \tag{2}$$

which is the same collineation as (1) expressed in line form. Now the fixed points and lines are found by putting $x'=x$ and $u'=u$. The collineations reduce to

$$\rho x_i = a_i x_i \quad \text{and} \quad \rho u_i = a_i u_i$$

which are identical in form. It follows that these equations will be consistent under the same conditions, viz. $\rho=a_i$, and that the coordinates of any point which satisfy the first set will be identical with the coordinates of a line that satisfy the second. The proof is valid for n dimensions by letting i range through $n+1$ values and replacing "line" by S_{n-1}, i.e. by a linear space of $n-1$ dimensions.

If the a's are distinct the collineation (1) [or (2)] has three fixed points and three fixed lines forming a triangle. But if two of the a's are equal, the collineation—which is then called a *homology*—has a point of fixed lines (the center) and a line of fixed points (the axis). The centers and axes of the homologies [type (*b*)] together with the fixed points and lines of the collineations of type (*a*) in a group may be called the *invariant configuration* of the group. As a consequence of the theorem proved above, we have at once:

The invariant configuration of a group is self-dual.

Thus every collineation group leads us to one or more associated configurations,—the invariant configuration and others which are frequently contained in it. These configurations include some of the most interesting to be found in geometry and, when studied in relation to the invariant curves of the groups, afford an opportunity for the contemplation of the highest form of beauty, a beauty whose full appreciation is the prerogative solely of the mathematician since much of the complete figure is usually imaginary.

The converse question is, does every configuration determine a finite group? Certain well known groups may be so defined. Thus the ternary octahedral G_{24} comprises those transformations which permute in all possible ways the vertices of a complete 4-point. The ternary Hesse G_{216} may be obtained by asking for those collineations which transform into itself the Hesse configuration—the configuration associated with the nine flexes of the plane cubic. Coble has shown that the collineations which transform into themselves the seven points and seven lines in finite geometry (mod 2) constitute a group of order 168, simply isomorphic with the Klein group in the general projective plane.[1]

[1] Two groups G and G' are simply isomorphic if (*a*) there is a (1, 1) correspondence between their elements, (*b*) such that the product of any two elements of G corresponds to the product of the two corresponding elements of G'. Two groups which are simply isomorphic are said to be abstractly identical.

The major problems that any collineation group offers for study are the structure of the group itself—the relations between the elements, the nature and variety of the subgroups, its isomorphisms with other groups, etc.—and the invariant configuration and invariant curves. The structure may of course be studied purely algebraically and the knowledge gained may then be applied to the geometry. But I think enough has been said to indicate that the geometry is a useful weapon of attack on the fortress of the group itself. In particular a knowledge of special sets of conjugate points may be utilized in the discovery of the conjugate subgroups associated with them.

§3. *Self-Projective Curves*

The determination of the types of groups in any dimension, as well as the study of their structure, may be considered an algebraic or group-theoretic problem. However the first solution of this problem in one dimension, the binary realm, was effected by Klein with the aid of geometric considerations. Indeed these groups, in the abstract, are identical with the groups of rotations of a sphere which carry the regular inscribed polyhedrons into themselves. The groups associated with the cube and the octahedron are the same, since diameters of the sphere through the midpoints of the faces of the cube cut the sphere in the vertices of a regular octahedron. The two figures are in a sense space duals, the cube having 6 faces, 8 vertices and 12 edges while the octahedron has 8 faces, 6 vertices and 12 edges. Thus vertices and faces correspond in this duality while an edge is self-dual. Similarly the regular inscribed dodecahedron with 12 faces, 20 vertices and 30 edges is dual to the regular icosahedron with 20 faces, 12 vertices and 30 edges,—both admitting the same group. The regular tetrahedron is self-dual since lines joining the vertices to the center cut the sphere in the vertices of a second, the counter tetrahedron. To complete the theory, Klein devised the regular *dihedron*, a two-faced "polyhedron" consisting of two coincident regular polygons inscribed in a great circle.

The groups of rotations take their names from the polyhedrons which they leave invariant and are sometimes called the regular body groups. If a dihedron with n vertices lies in an equatorial plane it will be carried into itself by a rotation of period n about the axis passing through the poles, generating a cyclic group of order n. If to this group we adjoin the n rotations of period 2 about lines of symmetry of the dihedron, we obtain the complete group of the dihedron, a G_{2n}. For the other regular bodies there are three possible positions of the axis of rotation (a) through a vertex, (b) through the mid-point of a face and (c) through the mid-points of opposite edges. Rotations (c) are of period 2 for all of the groups; and while types (a) and (b) vary with the groups, all of either class are of the same period for any particular group. In the case of the tetrahedron, (a) and (b) coincide, both being of period 3. The number of rotations of the several periods are easily counted and the orders of the groups determined. The five regular body groups then are:

Name and Order	*Associated Polyhedron(s)*
Cyclic G_n	Dihedron
Dihedral G_{2n}	Dihedron
Tetrahedral G_{12}	Tetrahedron
Octahedral G_{24}	Octahedron, cube
Icosahedral G_{60}	Icosahedron, dodecahedron.

While the regular bodies were known to the ancients, it remained for Klein to point out the varieties of groups associated with them and to prove in particular that they are simply isomorphic with types of collineation groups in the binary domain. It seems to have been Klein's peculiar genius to excel in recognizing these analogies in mathematical theories which constantly challenge the wonder of every mathematician. I know of no more striking manifestation of that genius than his achievement in reading in the properties of the familiar —one might almost say household objects—the regular bodies, all the group theory implicit in them, a theory latent since the time of Pythagoras, eluding the searching scrutiny of the mathematical world for two and a quarter milleniums.

To my mind this geometric approach to the analysis of the groups in the binary field, because of its concreteness, simplicity and elegance, no less than for its intuitive directness is incomparably superior to any other. What other important theory perchance lies dormant in the sombre chrysalis of elementary geometry, awaiting the fullness of time and the creative touch of some future Klein to call it into a glorified being?

In the plane, as previously indicated, there are two main divisions of groups, (*a*) those which leave a triangle invariant (monomial groups) and (*b*) those which do not. In particular we have five regular body groups—ternary groups simply isomorphic with the several binary groups.[1] These are all monomial except the ternary icosahedral group. There are six varieties of groups of type (*b*) in the plane: The icosahedral G_{60}, a G_{36}, a G_{72}, the Hesse G_{216}, the Klein G_{168} and the Valentiner G_{360}. The G_{36} is one of three conjugate subgroups of the G_{72} which in turn is a self-conjugate or invariant subgroup of the Hesse group.

We have already mentioned one geometric problem connected with every group—that of the invariant configuration. Two others relate to the invariant curves. The first is the determination of the *complete system* of invariants, i.e. a finite number of forms in terms of which all others may be rationally and integrally expressed. This complete system for groups in n homogeneous variables comprises a minimum of $n+1$ forms of which n are algebraically independent. This problem has been solved for all of the principal classes of groups.

The second problem is that of self-projective curves. Since a collineation defines analytically a projection, a curve which is invariant under a collinea-

[1] In every dimension there is at least one regular body group simply isomorphic with each of the binary group. Ciani, Annali di Matematica (1902), has discussed the different species in space.

tion group is termed self-projective. The problem may be formulated thus: (1) What varieties of groups may a curve of given order m admit? (2) what are the projectively distinct species of curves of order m invariant under the same group? This problem for curves restricted only by the group requirements has been solved for $m=2, 3, 4, 5, 6$ and for rational curves for $m=2, 3, 4, 5, 6, 7$. A conic may admit any of the regular body groups. The general cubic is invariant under a G_{18}, a subgroup of the G_{36} above; the harmonic cubic under the G_{36}; and the equianharmonic cubic $x^3+y^3+z^3=0$ under the subgroup G_{54} comprising all of the monomial transformations of the Hesse G_{216}. The general rational cubic admits a dihedral G_6 and the cuspidal cubic an infinite group.

A self-projective quartic is necessarily special but there are numerous interesting varieties. The octahedral G_{24} has a pencil of them

$$x^4 + y^4 + z^4 + 6\lambda(y^2z^2 + z^2x^2 + x^2y^2) = 0,$$

including a repeated conic ($\lambda=1/3$), the projective lemniscate ($\lambda=\infty$), two Clebsch curves[1] ($\lambda=1, -1/2$) and their associated Lüroth curves ($\lambda=-1/3$, $7/6$, respectively), a pair of conics[2] ($\lambda=-1/6$), a Dyck curve ($\lambda=0$) and a pair of Klein quartics ($\lambda=-\frac{1}{2}\pm\frac{1}{2}\sqrt{7}$). The Dyck and Klein quartics admit groups of orders 96 and 168 respectively.

The curves of lowest orders associated with the Hesse and Valentiner groups are sextics.

Self-projective curves are always interesting because of the remarkable way in which they adjust themselves to the rather severe demands of the groups. If a curve is invariant under a G_n, the points of the curve in general distribute themselves in sets of n conjugate points, though some may belong to special sets. In particular, the singularities will be found among the conjugate sets, special or general. The number of special sets of points on an invariant curve is limited, for the number of special sets under the group is finite unless the group contains homologies. In the latter case there will be an infinite number of special sets, lying on the axes of homology, but only a finite number of these can lie on the curve.

The simplest example of a self-projective curve is one admitting a group of order two, consisting of a reflexion and the identity. Even here the curve is considerably specialized. First of all, each intersection of the axis of reflexion with the curve is either a multiple point or a contact of a tangent from the center of reflexion. All other points—except the center, if it happens to lie on the curve—must fall into pairs harmonically separated from the center by the axis. All tangents from the center whose contacts do not fall on the axis must be

[1] See a paper by Arnold Emch, Bulletin of the American Mathematical Society vol. 35 (1929), p. 389. The first Lüroth quartic ($\lambda=-1/3$) degenerates into four lines, dual to the special set of four points conjugate under the group—see below.

[2] This pair of conics admit an enlarged group G_{48} of collineations and correlations, the correlations arising from the products of the elements of G_{24} by a polarity. See Wear, American Journal of Mathematics) vol. 42 (1920), p. 118.

multiple tangents with contacts harmonically paired. The contacts on the axis can arise only from the coincidence of an even number of points—they may be undulations, e.g., but not flexes. A multiple tangent from the center may have an odd number of contacts only if one fall on the axis. If the center lies on the curve, the least thing that can happen is that the point be an inflexion. It might be a flex of higher order, the tangent cutting out an odd number of coincident points. The center may be a biflecnode and on curves of even order it may be a tac-node. If a curve is restricted only by admitting a reflexion, the intersections of the axis will be contacts of simple tangents from the center, while other tangents from the center will be ordinary double lines. Again if the center lies on the curve, we normally expect it to be an ordinary inflexion if the curve is of odd order, a biflecnode if the curve is of even order. Now most groups contain several reflexions which comprise one or more conjugate sets. If k of these belong to such a conjugate set, the yield of geometrical fruitage is exactly k-fold, as illustrated by those curves possessing k-fold symmetry and invariant under a dihedral G_{2k}, k odd. If k is even the curve still has k-fold symmetry but the axes divide into two conjugates sets, the symmetry with respect to the two sets not being the same.

Suppose that a curve is invariant under a homology[1] of period three, e.g. $x'=\omega x, y'=y, z'=z, \omega^3=1$. Then lines through the center $(1, 0, 0)$ cut out triples of conjugate points. The intersections of the axis, $x=0$, thus will be flexes, cusps etc. whose tangents meet at the center—or they will be triple points, six-fold points etc. Other tangents from the center must be at least triple tangents. If the center is on the curve, it will be at least an undulation since it must arise from the coincidence of a minimum of four points. Thus the quartic curve $y(y+z)^3-x^3z=0$ admits a cyclic G_3 generated by the homology above. The center is an undulation (whose tangent is $z=0$) and the axis cuts out a triple point, and a flex whose tangent, $y=0$, passes through the center.

Another important principle is that two curves which are invariant under the same group must intersect in sets of conjugate points. Klein has utilized this principle in finding the complete system of invariants of a group.[2] It is also helpful in constructing sets of conjugate points on an invariant curve. There are ∞^2 sets of conjugate points associated with any group since any point in the plane gives rise to one set. Only a single infinity of these however will lie on an invariant curve. As an illustration, the equianharmonic cubic $x^3+y^3+z^3=0$ is invariant under a monomial G_{54}, generated by the homology of the preceding paragraph and the transformations

$$\begin{aligned} x' &= x, \quad z, \quad -x, \\ y' &= \omega y, \quad x, \quad -z, \\ z' &= \omega^2 z, \quad y, \quad -y. \end{aligned}$$

[1] The homology of course generates a cyclic group of order three which contains a second homology.

[2] See for example Fricke, *Lehrbuch der Algebra* (1926), vol. 2, p. 203.

The curve contains three special sets of points, (*a*) the 9 flexes, (*b*) the 27 sextactic points, (*c*) a set of 18 points, vertices of six Poncelet polygons, cut out by the invariant sextic $y^3z^3+z^3x^3+x^3y^3=0$. These as well as all general sets on the cubic are cut out by the pencil of invariant curves[1]

$$x^6y^6z^6 + \lambda(y^3z^3 + z^3x^3 + x^3y^3)^3 = 0.$$

The special sets are cut out multiply by particular curves—the flexes 6 times ($\lambda=0$), the 18 points 3 times ($\lambda=\infty$) and the sextactic points twice ($\lambda=4/27$).

A problem introduced into geometry by the analytic method is that of canonical forms, i.e. the discovery of the simplest form to which the equation of a locus may be reduced. From the point of view of pure geometry the problem is artificial since it merely concerns the optimum choice of the reference framework. If a curve is invariant under a group, the selection of the reference triangle is usually not difficult. We saw that the collineations in a group are of two kinds—those with a unique fixed triangle and the homologies with ∞^2 fixed triangles. Each of the latter has as one vertex and opposite side a center and axis of a homology. *When the curve is in canonical form the triangle of reference is some one of these triangles associated with the individual collineations*, i.e. *it is part of the invariant configuration of the group.*

Thus the equianharmonic cubic assumes the form given above when referred to the invariant triangle of the G_{54}. But there is another form entitled to consideration as a canonical form of the curve. For if we select as reference triangle any one of the six Poncelet triangles whose vertices comprise the special set of 18 conjugate points, the equation of the curve may be reduced to $x^2y+y^2z+z^2x=0$, or $xy^2+yz^2+zx^2=0$.

We shall close this section with a statement of some general theorems concerning self-projective curves.

1°. *The dual of any self-projective curve admits the same group.*

For suppose that any collineation of the group is reduced to the form (1), then the line equation of the collineation is (2). The inverse of (2) is

$$\rho u_i' = a_i u_i, \tag{3}$$

which is identical in form with (1). Hence if the point curve $f(x_1, x_2, x_3)=0$ is invariant under (1), the dual curve $f(u_1, u_2, u_3)=0$ is invariant under (3), which is merely some power of (1) in line form. Thus, since the projective lemniscate is an invariant of the ternary octahedral G_{24}, we infer that its dual the projective astroid is likewise an invariant.

2°. *If an invariant curve C_{2n}, of order 2n, factors into two curves C_n and C_n', neither of which is invariant under the whole group, C_n and C_n' are projectively equivalent*, since they must be interchanged by some elements of the group.

3°. *A C_n cannot admit a homology of period greater than n*, since conjugate points must lie in sets of *n* on lies through the center.

[1] The curves of this pencil are composite, each composed of three sextics.

4°. *If a C_n is invariant under a homology of period k, the first polar of C_n is composite, the axis splitting off $k-1$ times. The axis splits off from the second polar $k-2$ times, etc. If $k=n$, the first polar of an arbitrary point on the axis degenerates into $n-1$ lines passing through the center.*[1]

§4. *Self-Projective Rational Curves*

The group problem for rational curves differs from that for general algebraic curves because of the parametric representation that characterizes the former. By definition, the homogeneous coordinates of a point on a rational curve are expressible as rational, integral, algebraic functions of a single parameter,—i.e. as binary forms. Hence the geometry on a rational curve belongs to the binary domain. The following theorem is fundamental for rational curves.

1°. *If the points of a rational curve in S_{n-1} are transformed into themselves by an n-ary group, the parameters of the points are permuted by a binary group of the same order.*

The only groups admissible then are the regular body groups, binary and n-ary. Other general theorems for rational plane curves are:

2°. *If a rational curve of order m is invariant under a cyclic group of order n, $n>m$, it is invariant under a one-parameter group.*

3°. *A rational curve of order m, m odd, cannot admit the tetrahedral, octahedral nor icosahedral group, nor a dihedral G_{2k}, if k is even.*

We have already observed (§1) that the maximum order of a dihedral group is $2m$. Theorems 2° and 3° thus restrict the types of groups for curves of odd degree to cyclic groups of order n, $n=2, 3, \cdots, m$, and dihedral groups of order $2k$, $k=3, 5, \cdots, m$.

4°. *If a rational curve of order m is invariant under a cyclic group of order m (but not a dihedral G_{2m}) the group cannot contain a homology. The curve has a multiple point of order greater than two, having however but two distinct parameters.*[2]

We have in the plane then a partial parallelism between the binary and ternary groups associated with self-projective rational curves. Every element of the binary group has two fixed points (parameters). These points as well as their tangents are fixed under the ternary group, which however leaves at least one other point and line fixed. Hence there will be a difference in the special sets for the two groups. Every non-cyclic binary group has three special sets, corresponding to (a) the vertices, (b) the midpoints of the faces and (c) the midpoints of the edges. These will correspond to special sets of the same number of points under the ternary group if each parameter is attached to a single point. But if a binary set of $2k$ are parameters of k nodes, the nodes will form a special ternary set of k points although their tangents will be a special ternary set of $2k$ lines. These are the only special sets of points lying on an invariant

[1] This theorem is proved in a paper read at the Nashville meeting of the American Mathematical Society.

[2] These theorems are proved in a series of papers by the author on self-projective rational curves in the American Journal of Mathematics, vols. 36, 38, and 47.

curve but the ternary group contains an infinite number of special sets not on the curve. A binary cyclic group leaves each of two points fixed while the corresponding ternary group has at least one additional fixed point.

Perhaps these principles will be sufficiently illustrated by the binary and ternary octahedral groups and the invariant projective astroid, which in metrical form is the familiar "hypocycloid of four cusps," a rational curve of class 4 and order 6. For convenience we shall denote ternary groups by G.

The G_{24} is the group of the complete quadrangle, permuting the vertices in all possible ways. The diagonal triangle is fixed under the whole group. The vertices and sides of this triangle are centers and axes of three reflexions, which with the identity constitute a dihedral G_4, an invariant sub-group of G_{24}. Each vertex and opposite side are likewise the fixed point and line of one of the three conjugate dihedral subgroups G_8. The sides of the quadrangle are the axes of six other reflexions, conjugate under the group. Each vertex of the quandrangle is fixed under one of the four conjugate dihedral subgroups G_6. The four cyclic subgroups G_3 of the dihedral G_6's leave each two additional points unaltered, giving rise thus to a special set of eight points. The centers of the set of 6 reflexions lie in pairs on the sides of the invariant triangle—each pair harmonically separated by a pair of vertices of the triangle. On each side of the triangle is a third pair, harmonic both to the vertices and the centers on that side. We thus obtain a second set of six conjugate points. The special sets of points of G_{24} thus comprise:

A set of 3, vertices of the invariant triangle;
A set of 4, vertices of the invariant quadrangle;
A set of 6, centers of the set of six reflexions;
A counter set of 6, described above;
A set of 8, which with the set of four are the fixed points of the four cyclic G_3's;
∞^1 sets of 12, lying in pairs on the sides of the invariant quadrangle;
∞^1 sets of 12 lying in fours on the sides of the invariant triangle.
Dual to each of these is a special set of conjugate lines.
When written in the usual Cartesian form

$$x^{2/3} + y^{2/3} = a^{2/3}, \tag{4}$$

the curve exibits symmetry with respect to the origin and four equispaced lines through the origin, i.e. one of the dihedral G_8's is real. The invariant triangle of the group is then formed by the rectangular axes and the line at infinity. Since the invariant triangle is treated symmetrically by the group, we infer that the line at infinity is a third double-cusp tangent, the cusps falling at the circular points. The six cusps thus lie on a circle, the invariant conic of the group. The flexes all go to form the cusps, while the three double-cusp tangents are the only bitangents. There are however four nodes, isolated at $(\pm a/8^{1/2}, \pm a/8^{1/2})$, which are the vertices of the invariant quadrangle.

The binary group has but three special sets of points (parameters) (a) the cusps, the counter set of 6, corresponding to the octahedron vertces, (b) the 8

nodal parameters, corresponding to the cube vertices and (*c*) a set of 12, the residual intersections of the sides of the quadrangle of double points. Thus (*a*) and (*c*) are special sets of the ternary group as well while (*b*) determine a set of 4 in the ternary group.

If we define the curve as the envelope of a line which the rectangular axes cut in a segment of constant length a, the equation in Plücker coordinates is found at once to be

$$1/u^2 + 1/v^2 = a^2$$

the point equation of which is (4). The locus of the middle point of this line segment is a circle (conic on two cusps), having four proper contacts with the curve. This is a member of the invariant pencil of double-contact conics (concentric circles) belonging to the real dihedral G_8. Since the three dihedral G_8's are conjugate, the circle is one of a set of three conics conjugate under G_{24}. Bearing in mind the projective version of middle point of a segment and the geometry of a binary cubic, we have the theorem:

If the intersections of a variable tangent line of the projective astroid with the three double-cusp tangents are represented by a binary cubic, the three cubicovariant points will trace, each a conic passing through two cusps and having four contacts with the curve. The 12 contacts are a special set under both the binary and the ternary group.

THE ELDER ĀRYABHAṬA'S VALUE OF π

By SĀRADĀKĀNTA GĀṄGULI, Ravenshaw College, Cuttack, India

The credit of discovering two remarkably approximate values of π in ancient times belongs to two Asiatic scholars of the fifth century A.D. One of them is the Chinese astronomer Tsu Ch'ung-chih (born 430 A.D.) and the other is the Indian astronomer and mathematician Āryabhaṭa (born 476 A.D.) known as the elder Āryabhaṭa as distinguished from another astronomer-mathematician of the same name who lived in the tenth century A.D. Tsu gives the value 355/113 ($=3.141592\cdots$)[1] and Āryabhaṭa the value 62832/20000 ($=3.1416$). In point of accuracy Tsu's value is superior to Āryabhaṭa's value which, however, can claim practical advantages over the other. Tsu's value has not been the subject of much discussion and its authorship has not been assigned to a foreign writer or to a later Chinese investigator. But Āryabhaṭa's value has received an altogether different treatment. It has been the subject of a long controversy and its genuineness has been seriously questioned. Attempts have been made to attribute the elder Āryabhaṭa's value of π to an unknown Greek source or to the younger Āryabhaṭa.

[1] Mr. Kaye omits this value from his list published in *Indian Mathematics* (p. 33) but includes Tsu's inaccurate value, 22/7, in it.

The arguments urged in favour of a Greek origin are:

(i) According to Rodet "The choice of a diameter of *two myriads* or rather of the number *one myriad* for the radius is assuredly a strong argument in favour of a Greek origin."[1] For, in his opinion, the Greeks alone of all peoples made the myriad a numerical unit of the second order. Sir Thomas Heath also endorses this view.[2]

(ii) According to Alberuni, Pulisá employed a value of π equal to the elder Āryabhaṭa's value.[3] Mr. Kaye takes this Pulisá to be anterior to both the elder Āryabhaṭa and Tsu Ch'ung-chih.[4]

(iii) The value 3.1416 was known to the Alexandrian scholars whose works may well have reached India.[5]

(iv) "The Hindu mathematicians took various values of π and no writer among them seems to have been uniform in his usage."[6]

(i) The unsoundness of the first of the above arguments has been shown by the present writer[7] and Dr. Bibhūtibhūṣaṇ Datta.[8] Āryabhaṭa used the numerical unit *ayuta* which is equivalent to the Greek myriad but which has been in use in India from before the birth of the Greek civilization. I may add that Rodet's argument does not seem to have appealed to Marie who does not see to which Greek writer the elder Āryabhaṭa's value of π could be attributed but who admits, as we all admit, that there had existed, even before the fifth century A.D., intercourse between Greece and India.[9] But mere existence of intercourse can not prove India's indebtedness to Greece for a result which, so far as our present knowledge goes, the latter did not know.

(ii) The fallacy of this argument has been thoroughly exposed by Mr. Nalinbihāri Mitra[10] and Dr. Bibhūtibhūsan Datta.[11] Dr. Datta has shown that Alberuni's Pulisá is not the author of the original *Pulisá-siddhānta* or of the one known to Varāhamihira but of a recast of the same work with various changes and additions. The author of the original *Pulisá-siddhānta* was a contemporary of the elder Āryabhaṭa according to Weber while, according to Kern, the former preceded the latter by a century.[12] Alberuni's Pulisá refers to Varāhamihira

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[1] *Journal Asiatique*, vol. 13 (1879), p. 411.

[2] *History of Greek Mathematics*, vol. 1 (1921), p. 234.

[3] Alberuni's *India*, vol. 1, p. 168; vol. 2, p. 72 and p. 67, where the fraction 14/15 is a mistake for 14/25.

[4] *Indian Mathematics*, p. 33.

[5] D. E. Smith, *History of Mathematics*, vol. 2 (1925), pp. 308–309.

[6] Ibid, p. 308.

[7] Journal of the Bihar and Orissa Research Society for March, 1926, pp. 83 and 84.

[8] Journal of the Asiatic Society of Bengal, vol. 22 (1926), pp. 36–37; hereafter referred to as *JASB*.

[9] Marie, *Histoire des Sciences Mathématiques et Physiques*, vol. 2, p. 73.

[10] *Modern Review* (Calcutta), vol. 18, p. 160.

[11] *JASB*, vol. 22 (1926), pp. 27, 37, 38.

[12] *JASB*, vol. 4 (1908), p. 115.

twice[1] and not once as stated by Mr. Kaye,[2] uses the elder Āryabhaṭa's value of π, and also takes his measure (namely, 3438′) of the radius as the radius of the sphere of each planet.[3] For a Greek origin of the elder Āryabhaṭa's value of π Mr. Kaye finds it necessary to establish the priority of Alberuni's Pulisá to the elder Āryabhaṭa. Hence he wants to take away Pulisá's reference to Āryabhaṭa and Varāhamihira from the mouth of Alberuni's Pulisá and to put it into the mouth of Alberuni himself.[4] But he is silent with respect to a second reference to Varāhamihira by Alberuni's Pulisá. He then concludes that "the traditional order . . . Pulisa, Āryabhaṭa, Varāhamihira . . . may be accepted as correct." We do not object to this conclusion. Only we would extend Mr. Kaye's list as follows: Pulisá (whose $\pi = \sqrt{10}$), Āryabhaṭa, Varāhamihira, Alberuni's Pulisá (i.e., the author of Alberuni's *Pulisá-siddhānta* whose $\pi = 3.1416$). Here we have maintained the order adopted by Mr. Kaye.

(iii) Dr. D. E. Smith attributes the value 3.1416 to Ptolemy who takes 3 8′ 30″ as the approximate value of π. He writes: "Since 3 8′ 30″ = 3.1416, his (Ptolemy's) value was very satisfactory."[5] But 3 8′ 30″ is not equal to 3.1416 but to 3.1417 (correct to 4 places of decimals). It is really equal to 377/120 whereas Ārybahaṭa's value is equal to 3927/1250. So it is difficult to see how the elder Āryabhaṭa's value of π was known to the Alexandrian scholars.[6] Thus the possibility of obtaining this value from their works disappears altogether. Hence Dr. Smith's alternative suggestion[7] that this value may have been found independently in India is the only possible conclusion which stands to reason.

(iv) Mr. Kaye's table of different values of π (*Indian Mathematics*, pp. 32 and 33) is mainly responsible for this argument. It shows that he has failed to correctly understand some of the results of Indian mathematical and astronomical works. The elder Āryabhaṭa uses only one value of π, namely, 62832/20000. But the following additional values have been wrongly attributed to him:[8]

(1) 3 ; (2) 3393/1080 = 3.14166 ;

(3) 600/191 = 3.14136 · · · (4) 16/9 ;

(5) 1.7.

(1) In the verse immediately preceding the one giving the elder Āryabhaṭa's value of π he states that the chord of the sixth part of the circumference of a circle is equal to the radius. This shows that where accuracy is required the elder Āryabhaṭa cannot take π to be equal to 3. Mr. Kaye has left his readers in the dark as to his reasons for ascribing this value of π to the elder Āryabhaṭa.

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[1] Alberuni's *India*, vol. 1, p. 266; vol. 2, p. 70.
[2] *JASB*, vol. 4 (1908), p. 115.
[3] Alberuni's *India*, vol. 2, p. 69.
[4] *JASB*, vol. 4 (1908), p. 115.
[5] *History of Mathematics*, vol. 2, p. 308.
[6] Also see *JASB*, vol. 22 (1926), p. 38.
[7] *History of Mathematics*, vol. 2, p. 309.
[8] Also see *Indian Education*, vol. 8, p. 351; *JASB*, vol. 4 (1908), pp. 121, 122 foot-note.

(2) The value 3393/1080, which is equal to Ptolemy's value 377/120, has been attributed to Āryabhaṭa because it is stated by Alberuni, on the alleged authority of Brahmagupta,[1] that Āryabhaṭa fixed the circumference as 3393 and the diameter as 1080. Using the value 62832/20000 of π the circumference of a circle of diameter $1080 = (62832/20000) \times 1080 = 3392.928 = 3393$ (to the nearest integer). It will thus be seen that Āryabhaṭa did not use the value 3393/1080 which he does not even mention in his *Gaṇitapāda* (i.e., the chapter on mathematics). The unnatural character of Mr. Kaye's deduction of this value will be clear from the following question: If it be hereafter found that the elder Āryabhaṭa has found, to the nearest integer, the circumferences of 10,000 circles of different diameters, how many different values of π should be attributed to him, seeing that the ratio of the circumference obtained to the corresponding given diameter is different in each case?

(3) Mr. Kaye has followed the same unnatural method in attributing the value 600/191 to the elder Āryabhaṭa. He writes: "Āryabhaṭa sets the radius = 3438 minutes whence[2]

$$\pi = (2 \times 90 \times 60) \div 3438 = 600/191 (= 3{\cdot}14136 \cdots).$$"

How has Āryabhaṭa come to take the radius = 3438 minutes? He is the first investigator to introduce this innovation.[3] Rodet answers this question thus: $(10800'/1.1416) = 3437'.7 = 3438'$ nearly; and he concludes that the value 3.1416 of π was used. But Mr. Kaye observes:[4] "This is not ingenuous. We might replace this value by Ptolemy's value and then we should have $10800 \times (120/377) = 3437.66 = 3438$, nearly; and we might just as forcibly conclude that Ptolemy's value was used."[5] If Āryabhaṭa had not given a value of π, one would have been justified in suggesting that he might have used Ptolemy's value. Mr. Kaye's observation, therefore, is not only not ingenuous but it also betrays an excessive amount of anti-Indian bias.

(4) Mr. Kaye attributes the value 16/9 to the elder Āryabhaṭa because the latter gives the formula $(\pi r^2)^{3/2}$ for the volume of a sphere of radius r. This is based on the assumption that Āryabhaṭa also knew the volume to be $4\pi r^3/3$.

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[1] Alberuni's *India*, vol. 1, p. 168.

[2] *Indian Education*, vol. 8, p. 351.

[3] Mr. Kaye is not right when he says: "This new measure first occurs in the *Pulisá-siddhānta*, for Alberuni writes (i,275): 'Our calculation is based on this, that the *sinus totus* is 3438.' . . . . The source of this calculation . . . . is the *Pulisá-siddhānta*.' " In the last gap Alberuni has used the words "of Balabhadra." He says that he could not find anything of Āryabhata's works (*India*, i, 370). This is why he considers the *Pulisá-siddhānta* as the source of Balabhadra's calculation. It has already been stated that Alberuni's *Pulisā-siddhānta* refers once to Āryabhata and twice to Varāhamihira. In the original *Pulisá-siddhānta* or rather in the one known to Varāhamihira $\pi = \sqrt{10}$ and not 3·1416, and $r = 120$ and not 3438 (*Indian Mathematics*, p. 11).

[4] *JASB*, vol. 4 (1908), p. 125.

[5] Mr. Kaye then says: "Ptolemy's value was most probably used." Thus, according to Mr. Kaye, Āryabhata used both 600/191 (also see *Indian Mathematics*, p. 11) and 377/120 in deducing his measure (namely, 3438′) of the radius of a circle.

As there is no justification for this assumption—and Mr. Kaye is not prepared to give Āryabhaṭa credit for a correct expression for the volume of a sphere—, it is wrong to say that in practical applications Āryabhaṭa uses the value 16/9 for π.

(5) 1.7 seems to be a misprint for $1.\dot{7}$ which is the decimal equivalent of 16/9.

Hence it cannot be said that the elder Āryabhaṭa uses various values of π and that he is not uniform in his usage.

For rough calculations where rapidity is preferred to accuracy (e.g., in finding roughly the quantity of grain in a mound) Brahmagupta, Mahāvīra, Śrīdhara, the younger Āryabhaṭa, and Bhāskara take $\pi = 3$. For a greater degree of accuracy Brahmagupta, Mahāvīra, Śrīdhara, and the younger Āryabhaṭa use the value $\sqrt{10}$ and Bhāskara 22/7. For further accuracy the younger Āryabhaṭa employs[1] the value 22/7 and Bhāskara 3927/1250. To secure the same degree of accuracy the approximate value of $\sqrt{10}$ to be used must always depend on the magnitude of the number to be multiplied by $\sqrt{10}$. Failing to understand this, Mr. Kaye wrongly attributes the values[2] 22/7 and 721/228 to Brahmagupta and the value[3] 19/6 to Śrīdhara. For a similar reason the value 377/120 has been attributed to Bhāskara.

Mr. Kaye is perhaps inwardly conscious of the weakness of his arguments in favour of a Greek origin of the elder Āryabhaṭa's value of π. So it must be traced to some other non-Indian source. Hence Mr. Kaye writes that possibly the rule giving the value "properly belongs to a later writer, possibly to Āryabhaṭa the younger."[4] If he could establish this point, the credit of discovering this value of π would go to the Arabic mathematician Mohammed Ben Musa who, Mr. Kaye dogmatically asserts,[5] did not copy it from the Hindus.

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[1] Dr. Bibhūtibhūsan Datta infers from the *Mahāsiddhānta*, chapter 16, verse 37, that the younger Āryabhaṭa employed another value of π, namely, 21600/6876(*JASB*, vol. 22,1926, p. 30.) But here the younger Āryabhata gives 6876/21600 or 191/600 as the value of $1/\pi$ which is required in finding the diameter of a circle when the circumference is given. As π is incommensurable, its approximate value is not the same as the reciprocal of the approximate value of $1/\pi$. He finds the circumference by multiplying the diameter by $\sqrt{10}$ or 22/7 and not by 21600/6876.

[2] Mr. Kaye attributes this value to Brahmagupta on the authority of Alberuni who states that Brahmagupta has taken $\pi = \sqrt{10}$ because it is nearly equal to $3\frac{1}{7}$ (*India*, vol. 1, p. 168). If Brahmagupta had been aware of the value $3\frac{1}{7}$, he would not have rejected it in favour of $\sqrt{10}$ which is not only more inconvenient in application but is also not so accurate as $3\frac{1}{7}$.

[3] Śrīdhara gives $\frac{1}{2}d^3[1+(1/18)]$ as the volume of a sphere of diameter d. From this Mr. Kaye infers that Śrīdhara uses the value $3\frac{1}{8}$ of π. The danger of deducing, from results obtained by using certain values of π, other values not actually used by the authors of the results has already been shown. Here also Śridhara uses $\sqrt{10}$ for π as shown below:

$$V = \frac{4}{3}\pi r^3 = \frac{d^3}{2}\frac{\pi}{3} = \frac{d^3}{2}\frac{\sqrt{10}}{3} = \frac{d^3}{2}\left(1+\frac{1}{9}\right)^{1/2} = \frac{d^3}{2}\left(1+\frac{1}{18}\right),$$

nearly.

[4] *Indian Education*, vol. 8, p. 350.

[5] *JASB*, vol. 4, (1908), p. 122.

Mr. Kaye's arguments attributing the elder Āryabhaṭa's value of π to a later Indian writer are as follows:

(a)[1] It is rather extraordinary that the elder Āryabhaṭa himself never utilised this value (*Indian Mathematics*, p. 12; *JRAS*, July, 1910, p. 754).

(b)[1] It was not used by any other Indian mathematician before the twelfth century (*Indian Mathematics*, pp. 12 and 13; *Indian Education*, vol. 8, p. 351; *JRAS*, 1910, p. 754).

(c) No early Indian writer quotes Āryabhaṭa as recording this value (*Indian Mathematics*, p. 13; *Indian Education*, vol. 8, p. 350; *JASB*, vol. 4, 1908, p. 122).

(d) "Āryabhaṭa who is supposed to have discovered, or, at least, introduced the most accurate value of π known in ancient times," gives a wrong rule for the volume of a sphere (*Indian Education*, vol. 8, p. 351).

The first three of the above arguments have been discussed by Dr. Bibhūtibhūṣaṇ Datta who has shown them to be false.[2] The elder Āryabhaṭa has used his value of π in preparing his table of sines and in obtaining 1080 *yojanas* as the measure of the diameter of the earth's wind whose circumference is 3393 *yojanas* and 1050 *yojanas* as the diameter of the earth whose circumference he takes to be 3299 *yojanas*. This value of π was employed also by Varāhamihira, Lalla, the author of the recasted *Pulisá-siddhānta*, and Utpala Bhaṭṭa, all of whom were posterior to the elder Āryabhaṭa and anterior to Alberuni (first half of the 11th century).

The fourth argument need not be taken seriously. A similar remark could be made against many authorities. Rāmānujan has given many incorrect formulae. Yet credit for discovering important results has not been denied him. The fact that Diophantus could not detect the obvious solution ($x=1$) of the equations

$$52x^2 + 12 = \text{a square}, \quad (\textit{Arithmetica}, \text{III.10}),$$

$$266x^2 - 10 = \text{a square} \quad (\textit{Arithmetica}, \text{III.11}),$$

does not stand in Mr. Kaye's way of giving him credit[3] for a solution of the equation $ax^2-b=y^2$, although Diophantus himself admits his inability to solve the equation, $15m^2-36=$a square.[4] Sir Thomas Heath's failure to detect the obvious root 7 of the equation:

$$p^4 - 2p^2 + 197 = \text{a square}^5$$

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[1] Being misled by Mr. Kaye's writings, Dr. Cajori also has advanced these two arguments (*A History of Mathematics*, 1922, p. 87).

[2] *JASB*, vol. 22 (1926), pp. 26, 27, 34, 35, 36, 39. The second and third arguments have been discussed also by Mr. Nalinbihari Mitra (*Modern Review*, vol. 18, pp. 160 and 161).

[3] *East and West* (Simla), vol. 17, p. 677, footnote.

[4] *Arithmetica*, VI. 14.

[5] *Diophantus* (2nd edition), p. 84, foot-note. This equation may be written as

$$\left(\frac{p^2-1}{2}\right)^2 + 7^2 = \text{a square},$$

whence $(p^2-1)/2=24$, or $p=7$.

does not prevent Mr. Kaye from citing him as an authority. Why, then, should a few incorrect rules given by the elder Āryabhaṭa go to prove his inability to give his value of π?

An examination of the verse giving the value 62832/20000 of π strongly suggests that the author of the verse must be anterior to Varāhamihira who, so far as our present knowledge goes, is undoubtedly the first to introduce the popular custom of expressing numbers in the word-symbol notation based on the principle of place-value. This custom has since been invariably followed by his successors with the exception of the younger Āryabhaṭa who uses his *kaṭapayādi* system of notation in the astronomical chapters of his *Mahāsiddhānta* and the popular word-symbol notation in the arithmetical portion of the work.[1] In the verse under consideration the complicated number 62832, instead of being given in the word-symbol notation as *rada-vasu-kara-rasa* (*rada* = 32, *vasu* = 8, *kara* = 2, *rasa* = 6) or as some other similar equivalent expression, has been stated as follows: Eight times one-hundred-and-four (literally, one hundred increased by four) and sixty-two thousand.

It will appear from the foregoing discussion that, in the absence of fresh evidence to the contrary, the credit of discovering the value 62832/20000 (= 3.1416) of π must, in all fairness, go to the elder Āryabhaṭa.

We do not know how Āryabhaṭa obtained this value of π. He might have got it by actual measurement of the circumference of a circle of radius 10000 or by calculation of the length of the perimeter of a regular polygon of a very large number of sides inscribed in such a circle.

QUESTIONS AND DISCUSSIONS

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSION

I. The Pascal Triangle

By Norman Anning, University of Michigan

It is a tradition that the diagrams for certain proofs of the theorem of Pythagoras need no explanation except "Behold!". In the opinion of the writer most teachers treat Pascal's triangle as something of the same sort. It is presented about like this: "Here is a pretty way of remembering the coefficients of the expansion of $(1+x)^n$ when n is a small positive integer. Observe how the numbers in any line may be readily obtained from those in the line

[1] This fact alone is sufficient to distinguish the *Mahāsiddhanta* of the younger Ārayabhata from the *Āryabhaṭīyam* of the elder Āryabhata. Dr. D. E. Smith is not right when he says that it is not possible as yet to differentiate clearly between the works of the two Āryabhatas (*History of Mathematics*, vol. 1, p. 156).

above. There is nothing to prevent you from extending the scheme as far as you wish. We call it Pascal's triangle although it was known centuries before his time. *Now* let us turn to the method discovered by Newton for making such a triangle unnecessary." Correct; it is a memory-helper and while the ancients regarded it as something almost magic there is no reason why we should. But there is more that can be said. The purpose of this note is to point out some of the ways in which it can be used to set a spark to student interest.

Inspection of it yields immediately the answers to many problems involving selection. Thus, properly to decide relative ability, two teams will require one game; three teams, three games; six teams, fifteen games; ten teams, forty-five games; etc.

As a second example take the well-known problem: In how many ways can I, without lost motion, go from home to church, j blocks west and k blocks south? The Pascal triangle supplies a table of answers for all values of j and k.

The symmetry of the triangle illustrates a theorem in combinations: Out of n eligibles there are as many ways of choosing k as of choosing $(n-k)$.

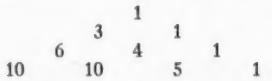
In the way the larger numbers are bunched toward the middle of any row the student of statistics has a symbol for his conviction that any measurable physical or mental characteristic possessed by a multitude of people will probably be found to be distributed among them so that the great majority will be close to the average of the group while a few will be noticeably above or below the average.

Suppose the triangle is placed as it appears as part of the large diagram. If we take from any horizontal line a fragment which begins or ends with 1, we can put with the numbers the words *point, straight line, plane, space,* (in this or in the reverse order) along with necessary grammatical connectives to compose the statements of true geometrical theorems. In the examples which follow, "line" means straight line, and "space" means the ordinary, flat, three-dimensional, ante-Einstein space in which all our experiences have been collected. In one plane two lines intersect in one point. In one space two planes intersect in one line. Through one point two lines determine one plane. In one space four planes determine six lines which intersect in four points.[1]

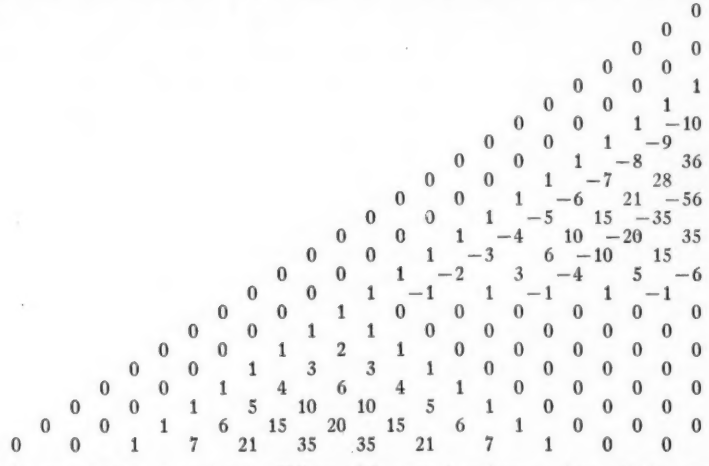
As a last example consider the extract which follows. It is the frame work for the complete description of a figure which is called Desargues' configuration. Translated into English, it reads: In space five planes intersect in ten lines which concur in ten points; there are three points on every line and six points on four lines on every plane. Any plane picture of the lines and points of this

[1] Readers who detect something rough and unfamiliar in the wording should be reminded that four given planes are not necessarily in the same space. If we want to have even two planes in the same space we are obliged to say so. In order to determine two general planes we require six points, and six points, if they are independent, determine a *five*-dimensional space. In five-space two given planes will usually have no point in common; in four-space two given planes will usually have just one point in common.

configuration is an answer to the question: How would you plant ten trees in ten rows, three in each row?



The triangle is extensible upward as well as downward and the upper part is as interesting as the lower.[1] Sir Oliver Lodge says in *Easy Mathematics* (Macmillan and Co., London, 1917, p. 334): "The binomial coefficients can be obtained by interjecting a single one into the middle of a row of noughts and then adding adjacent terms to make a term of the next series, as thus · · · ". His figure is the usual one (isosceles triangle of numbers on a horizontal base) but filled up with zeros at the right and left to the level of the interjected one. Now the word "interject" suggests the fiat of an external power. Let us assume, instead, that Sir Oliver's top row, · · · 0 0 0 1 0 0 0 · · · , arises as a result of the same law of progress which produces all the rows below it. Since 1 in every other place that we find it has been produced by the union of 0 and 1, it is natural that the 1 at the vertex should be so produced. So above it to left



and right we place a 0 and a 1. When this step has been taken all the rest that is shown in the figure follows naturally and need not be explained in detail. Notice that every place has been filled so that the chosen law of progress has been nowhere transgressed.

Reading in the horizontal rows we have now the coefficients for all integral powers of a binomial, negative as well as positive. Taking rows running from lower left to upper right we have, in successive rows under the last complete row of zeros, the so-called figurate numbers. These will not be defined or ex-

[1] AUTHOR'S POSTSCRIPT: A figure of the extended triangle is given by Edouard Lucas on p. 22 of his *Théorie des Nombres*.

plained here but it may be mentioned that their definition provides for the wedge of zeros in the place that it occupies. In rows running from upper left to lower right will be found coefficients of the expansion of $(1-x)^n$.

II. The Theory of Least Squares by Vectors

By J. P. Ballantine, University of Washington

In a recent note[1] we showed how Cramer's Rule for the solution of n equations in n unknowns, i.e., the determinantal formula, could be derived graphically. This was done by considering the columns of coefficients as the components of vectors.

When there are more equations than unknowns, a solution is, in general, not possible. The principle of least squares leads to the formation of so called normal equations, one equation for each unknown. The least square values of the unknowns are then obtained by applying Cramer's Rule to the normal equations.

In the present paper, we apply the methods of the previous note to the present situation. We derive the normal equations without recourse to the calculus.

For definiteness, suppose the system of equations to be:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= a_{14}, \\ a_{21}x_1 + a_{22}x_2 &= a_{24}, \\ a_{31}x_1 + a_{32}x_2 &= a_{34}. \end{aligned} \tag{1}$$

Denote the vectors (a_{1i}, a_{2i}, a_{3i}) by A_i for $i=1, 2, 4$. Equations (1) reduce to the single vector equation:

$$A_1x_1 + A_2x_2 = A_4. \tag{2}$$

It is now required to express A_4 as a linear combination of the two vectors A_1, A_2. Since the vector on the left of equation (2) is, for all values of x_1 and x_2 coplaner with A_1 and A_2, a solution of (2) does not in general exist. At this point the principle of least squares states (translated into the present vector notation) that the most satisfactory values of x_1 and x_2 to take are those which make the the vector

$$-A_1x_1 - A_2x_2 + A_4 \tag{3}$$

the shortest. The pair of values for x_1 and x_2 is called the least square solution.

Let A_3 denote any vector parallel to the vector (3), so that (3) is some arbitrary multiple A_3x_3 of A_3. For A_3x_3 to be shortest a necessary (provided $x_3 \neq 0$) and sufficient condition is that A_3 be perpendicular to the plane of the vectors A_1 and A_2. Hence A_3 must be perpendicular to the vectors A_1 and A_2, and we have the equations:

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[1] This Monthly, vol. 36 (1929), pp. 439–441.

$$A_3 \cdot A_1 = a_{13}a_{11} + a_{23}a_{21} + a_{33}a_{31} = 0,$$
$$A_3 \cdot A_2 = a_{13}a_{12} + a_{23}a_{22} + a_{33}a_{32} = 0. \tag{4}$$

From the definition of A_3, we have the equation

$$A_1x_1 + A_2x_2 + A_3x_3 = A_4. \tag{5}$$

Equations of components of (5) are

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= a_{14}, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= a_{24}, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= a_{34}. \end{aligned} \tag{6}$$

One possible procedure is to determine the components a_{13}, a_{23}, a_{33} of vector A_3 (except for a factor of proportionality) from (4), and then solve (6) for x_1 and x_2.

A better procedure, however, is to use equations (4) to eliminate x_3 from equations (6). Multiply equations (6) by a_{11}, a_{21}, a_{31}, respectively, and also by a_{12}, a_{22}, a_{32}, and add. The two resulting equations by (4) are free of x_3, and are the well known normal equations. Their solution is the least square solution.

I. Angular Velocity Determined by the Accelerations of Three Points

By E. L. Rees, University of Kentucky

In the December 1916 issue[1] of this "Monthly" Professors Peter Field and Alexander Ziwet discussed briefly the problem of determining the angular velocity of a rigid body given the accelerations of three of its noncollinear points. It was there indicated that the problem has four solutions. It is the purpose of this supplementary note to find these solutions in terms of the given accelerations and to discuss the various special cases.

Denoting by $\boldsymbol{p}$ and $\boldsymbol{q}$ the vectors from one of these points to the other two we have as our assumptions of rigidity,

$$\boldsymbol{p}^2 = \text{const.}, \quad \boldsymbol{q}^2 = \text{const.}, \quad \boldsymbol{p} \cdot \boldsymbol{q} = \text{const.} \tag{1}$$

The points being non-collinear we have also $\boldsymbol{p} \times \boldsymbol{q} \neq 0$.

Differentiating equations (1) twice with respect to the time, we find

$$\boldsymbol{p} \cdot \ddot{\boldsymbol{p}} = -\dot{\boldsymbol{p}}^2, \quad \boldsymbol{q} \cdot \ddot{\boldsymbol{q}} = -\dot{\boldsymbol{q}}^2, \quad \boldsymbol{p} \cdot \ddot{\boldsymbol{q}} + \ddot{\boldsymbol{p}} \cdot \boldsymbol{q} = -2\dot{\boldsymbol{p}} \cdot \dot{\boldsymbol{q}}. \tag{2}$$

Since the accelerations of the three points are given, $\ddot{\boldsymbol{p}}$ and $\ddot{\boldsymbol{q}}$ are known and therefore $\dot{\boldsymbol{p}}^2$, $\dot{\boldsymbol{q}}^2$, and $\dot{\boldsymbol{p}} \cdot \dot{\boldsymbol{q}}$ are also known.

Differentiating the equations $\dot{\boldsymbol{p}} = \boldsymbol{w} \times \boldsymbol{p}$ and $\dot{\boldsymbol{q}} = \boldsymbol{w} \times \boldsymbol{q}$, we have

$$\ddot{\boldsymbol{p}} = \dot{\boldsymbol{w}} \times \boldsymbol{p} + \boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{p}), \quad \ddot{\boldsymbol{q}} = \dot{\boldsymbol{w}} \times \boldsymbol{q} + \boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{q}). \tag{3}$$

[1] Vol. 23, pp. 371–381.

Eliminating $\dot{w}$ by using $p\cdot$ and $q\cdot$ as multipliers, we obtain

$$p\cdot\ddot{p} = w\times(w\times p)\cdot p = (w\cdot p)^2 - w^2p^2,$$

$$q\cdot\ddot{q} = w\times(w\times q)\cdot q = (w\cdot q)^2 - w^2q^2, \tag{4}$$

$$p\cdot\ddot{q} + \ddot{p}\cdot q = 2(w\cdot pw\cdot q - p\cdot qw^2).$$

Finally, eliminating $w\cdot p$ and $w\cdot q$ from equations (4), and replacing $p\cdot\ddot{p}$, $q\cdot\ddot{q}$, $p\cdot\ddot{q}+\ddot{p}\cdot q$ by $-\dot{p}^2$, $-\dot{q}^2$, $-2\dot{p}\cdot\dot{q}$, respectively, we get

$$(p\times q)^2w^4 - (p^2\dot{q}^2 + \dot{p}^2q^2 - 2p\cdot q\dot{p}\cdot\dot{q})w^2 + (\dot{p}\times\dot{q})^2 = 0, \tag{5}$$

from which we may find the scalar value of w. Let the coefficients in this equation be denoted by A, B and C respectively. Then

$$A > 0,\ B = p^2\dot{q}^2 + \dot{p}^2q^2 - 2p\cdot q\dot{p}\cdot\dot{q}$$

$$\geqq p^2\dot{q} + \dot{p}q - 2|p||q||\dot{p}||\dot{q}| \geqq 0,\ c \geqq 0.$$

The coefficient B equals zero only if $\dot{p}^2 = q^2 = 0$, *i.e.*, if $p\cdot\ddot{p} = q\cdot\ddot{q} = 0$.

It follows then that the roots of this quadratic in w^2 are both positive (or zero) if they are real, *i.e.*, if $B \geqq 2\sqrt{(AC)}$ (radical positive). We shall denote these roots by w_1^2 and w_2^2 and assume $w_1^2 \geqq w_2^2$.

From these inequalities and equations (2) it follows that the accelerations cannot be arbitrarily prescribed, but must satisfy the following conditions:

(a) $p\cdot\ddot{p} \leqq 0,\ q\cdot\ddot{q} \leqq 0,$ (b) $c \geqq 0$, i.e., $4p\cdot\ddot{p}q\cdot\ddot{q} \geqq (p\cdot\ddot{q} + \ddot{p}\cdot q)^2,$

(c) $B \geqq 2\sqrt{(AC)}.$

It may be shown that $B = 2\sqrt{(AC)}$ if the accelerations satisfy the simple conditions $\dot{p}^2/p^2 = \dot{q}^2/q^2 = \dot{p}\cdot\dot{q}/p\cdot q$, the numerators being the functions of $\ddot{p}$ and $\ddot{q}$ defined by equations (2).

We shall consider first the general case $B > 2\sqrt{(AC)}$ and $C > 0$. From equations (2) and (4) we have

$$w\cdot p = \pm\sqrt{(w^2p^2 - \dot{p}^2)},\ w\cdot q = \pm\sqrt{(w^2q^2 - \dot{q}^2)}, \tag{6}$$

$$[\pm\sqrt{(w^2p^2 - \dot{p}^2)}][\pm\sqrt{(w^2q^2 - \dot{q}^2)}] = p\cdot qw^2 - \dot{p}\cdot\dot{q},$$

the signs of the radicals being chosen so as to satisfy the last equation. Obviously there are two ways of choosing the signs for any given w.

Consider the equations

$$r\cdot p = \pm\sqrt{(w^2p^2 - \dot{p}^2)},\ r\cdot q = \pm\sqrt{(w^2q^2 - \dot{q}^2)}, \tag{7}$$

where w^2 is a root of equation (5) and r is the running coordinate. Each of these equations represents two planes. Hence these equations, treated simultaneously, with the proper association of signs, represent two lines symmetrically situated with respect to the origin and perpendicular to the plane of the three points.

If these lines intersect the sphere $r^2=w^2$, the vectors terminating in the four points of intersection will be the vectors representing the angular velocities. These vectors will lie along two intersecting lines, two of them being the negatives of the other two.

We shall see that the lines

$$r\cdot p = \pm\sqrt{(w_1^2p^2-\dot{p}^2)}, \quad r\cdot q = \pm\sqrt{(w_1^2q^2-\dot{q}^2)}$$

intersect the sphere $r^2=w_1^2$, and we shall find explicit expressions for the w's We shall also see that w_2^2 cannot be used since it leads to an imaginary result.

Letting P denote the perpendicular to one of these lines, we may write $P=lp+mq$, and squaring, $P^2=l^2p^2+2lmp\cdot q+m^2q^2$. Replacing r in equations (7) by this expression for P, we get

$$(8)\quad lp^2+mp\cdot q=\pm\sqrt{(w^2p^2-\dot{p}^2)}, \quad lp\cdot q+mq^2=\pm\sqrt{(w^2q^2-\dot{q}^2)},$$

from which we may find l and m, and therefore P.

To get P^2 without the laborious reductions entailed by direct substitution, we multiply the first and second of these equations by l and m respectively, and add. Then substituting the values of l and m found above, and simplifying by making use of (6), we find

$$P^2=\frac{-p^2\dot{q}^2-\dot{p}^2q^2+2p\cdot q\dot{p}\cdot\dot{q}}{p^2q^2-(p\cdot q)^2}+2w^2=-\frac{B}{A}+2w^2.$$

Indicating by subscripts the corresponding P's and w's ,we have

$$P_1^2=-(w_1^2+w_2^2)+2w_1^2=w_1^2-w_2^2, \quad P_2^2=-(w_1^2+w_2^2)+2w_2^2=w_2^2-w_1^2.$$

Since $w_1^2>w_2^2$, the above equations show that the pair of lines corresponding to w_1 will intersect the sphere $r^2=w_1^2$, and that P_2^2 must be discarded since it is negative.

From a figure which can easily be drawn, it can be seen that $w_1=P_1+np\times q$, where n is an unknown scalar. To determine n, we square and get

$$w_1^2=P_1^2+2[P_1pq]n+(p\times q)^2n^2.$$

But $[P_1pq]=0$ since the vectors are coplanar. Therefore

$$n=\pm\sqrt{\left[\frac{W_1^2-P_1^2}{(p\times q)^2}\right]}=\pm\sqrt{\left[\frac{W_2^2}{(p\times q)^2}\right]}$$

$$=\pm\frac{|W_2|}{|p\times q|}, \text{ and } W_1=P_1\pm|W_2|(p\times q)_1,$$

or, if the other signs of the radicals in (7) are used,

$$W_1=-P_1\pm|w_2|(p\times q)_1,$$

$(p\times q)_1$ indicating a unit vector.

Thus we obtain the four values of w.

We thus see that when the accelerations of three non-collinear points of a rigid body are given, subject to the conditions stated above, the magnitude of the angular velocity of the body is determined uniquely, and that the angular velocity may have any one of four directions.

We shall now consider the special cases:

I. The case in which $B>2\sqrt{(AC)}$, $C=0$. In this case one root w_2^2 of equation (5) vanishes. Hence $\boldsymbol{w}_1=\pm\boldsymbol{P}_1$, *i.e.*, the lines are tangent to the sphere and the vectors representing the angular velecities are the radii drawn to the points of tangency. There are thus only two solutions and the $\boldsymbol{w}$'s coincide with the perpendiculars to the two lines, and are therefore in the plane of $\boldsymbol{p}$ and $\boldsymbol{q}$; *i.e.*, the plane of the three points whose accelerations are given. $\boldsymbol{w}$ is a scalar multiple of $\boldsymbol{p}$ if $\ddot{\boldsymbol{p}}=0$, or a scalar multiple of $\boldsymbol{q}$ if $\ddot{\boldsymbol{q}}=0$.

II. The case in which $B=\sqrt{(AC)}$.

(a) $B>0$ ($\boldsymbol{p}\cdot\ddot{\boldsymbol{p}}$ and $\boldsymbol{q}\cdot\ddot{\boldsymbol{q}}$ are not both zero). Here $w_1^2=w_2^2$ and $P_1^2=P_2^2=0$. This shows that the lines pass through the origin and since they are perpendicular to the plane of $\boldsymbol{p}$ and $\boldsymbol{q}$ they are coincident. Hence there will be only two piercing points, and the two values of $\boldsymbol{w}$ are given by

$$\boldsymbol{W}=\pm\,|\boldsymbol{w}|\,(\boldsymbol{p}\times\boldsymbol{q})_1.$$

We conclude that the angular velocity is perpendicular to the plane of the three given points, but that its sense is indeterminate. The velocities $\dot{\boldsymbol{p}}$ and $\dot{\boldsymbol{q}}$ are coplanar with the three points.

(b) $B=0$ ($\boldsymbol{p}\cdot\ddot{\boldsymbol{p}}=\boldsymbol{q}\cdot\ddot{\boldsymbol{q}}=0$). In this case C will also vanish and the roots of equation (5) will be zero. Hence there will be no rotation.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of New York.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Wells, Webster, and Hart, W. W. *Modern Algebra, Third Semester Course.* New York, D. C. Heath and Company, 1929. viii+266 pages.

Features of this secondary school text are: "diagnostic tests and remedial instruction" in the early review chapters; a chapter on functional relationship; complete treatment of quadratic equations; the elements of trigonometry.

Kellogg, O. D. *Foundations of Potential Theory.* Berlin, Julius Springer, 1929. x+384 pages.

". . . . A two-fold purpose: first, to serve as a introduction for students whose attainments in the calculus include some knowledge of partial derivatives and multiple integrals; and secondly, to provide the reader with the fundamentals of the subject, so that he may proceed immediately to the applications, or to the periodical literature of the day."

Granville, W. A. *Elements of the Differential and Integral Calculus.* Revised by P. F. Smith and W. R. Longley. Boston, Ginn and Company, 1929. xii+516 pages. $3.20.

Some rearrangements of topics have been made; some topics in the older edition have been omitted, and replaced by other topics of importance and interest.

Dickson, L. E. *Introduction to the Theory of Numbers.* University of Chicago Press, 1929. ix+183 pages.

"The aim of this book is not technique, but the central ideas of the subject . . . . . The book is intended for beginners and develops the subject from first principles. [The subjects treated are] divisibility, congruences, quadratic residues, and the reciprocity law; binary quadratic forms; and Diophantine equations."

Kaiser, Ludwig. *Über die Verhältniszahl des Goldenen Schnitts, die Reihe der mit ihr zusammenhängenden ganzen Zahlen und eine aus dieser abgeleitete Reihe.* Leipzig, B. G. Teubner. viii+124 pages.

Lainé, E. *Premières lecons de géomètrie analytique et géomètrie vectorielle.* Paris, Librairie Vuibert, 1929. 48 pages.

Bouligand, Georges. *Complements et exercises sur la mécanique des solides.* Paris, Librairie Vuibert, 1929. viii+132 pages.

Vogel, Kurt. *Die Grundlagen der Ägyptischen Arithmetik, in ihrem zusammenhang mit der 2: n Tabelle des Papyrus Rhind.* München, Michael Beckstein, 1929. vi+212 pages.

Smith, David Eugene. *A Source Book in Mathematics*: Source Books in the History of the Sciences, Gregory D. Walcott, General Editor. New York, McGraw-Hill Book Company, 1929. xxviii+702 pages. $5.00.

Walther, Alwin. *Einführung in die Mathematische Behandlung Naturwissenschaftlicher Fragen.* Erstes Teil, Funktion und Graphische Darstellung, Differential- und Integralrechnung. Berlin, Julius Springer, 1928. viii+220 pages.

A treatise on analytic geometry and the calculus, intended primarily for students of the sciences. Some attention is given to monography, and physical applications are stressed throughout.

Walther, Alwin. *Begriff und Anwendungen des Differentials, mit besonderer Berücksichtigung der Bedürfnisse des Unterrichts und der Naturwissenschaften.* Leipzig, B. G. Teubner, 1929. iv+96 pages.

A similar but briefer work than the foregoing.

REVIEWS

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By E. W. Hobson. Vol. I, xvi+671 pp., 1921; Vol. II, x+780 pp., 1926. Second Edition. Cambridge University Press.

Hobson's *Theory of Functions* is a unique work, taking its character from the author's lifelong, scholarly, critical and comprehensive study of those parts

of analysis that deal with real functions. As is well known, no other book in this field attempts to present with such detail comparably as many topics. The book is exceptional, too, in having the dual nature of a treatise and a report; much of it is a record of systematizations and extensions attained through long meditation and personal research, while other parts are reëxpositions, not always organically worked in with the rest of the material, of recent journal articles. Thus the work of Hobson, unlike such books as the *Grundzüge* of Hausdorff or the *Reelle Funktionen* of Carathéodory, for example, which are written in a sustained mood, lacks the unity and finish one expects in a treatise or the artistic flow of a *Cours d'Analyse*. This is especially true of the second edition, which undertakes the difficult task of presenting in book form a great diversity of topics repeatedly subject to change of treatment on account of intensive modern research. On this account, too, the work has something of the stimulus which material arouses that has not yet been fashioned in final form.

Throughout there is the somewhat leisurely and truly British style of one who has more than the interest of the narrow specialist; in the manner of a natural philosopher, Hobson is intent on giving the reader the broad scientific bearings of the content. Illuminating historical sidelights, too, come in from time time. While this philosophical point of view is justly impressive, one misses in the occasional "metamathematical" remarks the intuitive insight or profound familiarity which are often the mark of those who, like Zermelo, read little but live largely with ideas coming from within. Thus in reference to the controversies in point sets, Hobson adopts a reasonable middle course; but his own comments on controversial questions are wanting somewhat in incisiveness and conviction.

After the introduction of irrational numbers according to the theories of Dedekind and Cantor, one passes to the descriptive and metric properties of point sets, (the former centering around the notion of limit and the latter around that of content and measure) cardinal number, and order type. The interest here is partly in the subject matter itself and partly in supplying the requisite preliminaries for the theory proper of real functions, which opens with Chap. V, Vol. I. p. 256. In this chapter are considered various types of continuity (for example, uniform, absolute, semi-series, and approximate continuity) functions of limited variation, properties of derivatives, including the comprehensive results of Denjoy and Young, and other things. Chaps. VI, VII and VIII of Vol. I are devoted to Riemann, Lebesgue, and non-absolutely convergent integrals, respectively, the last chapter featuring the Denjoy integral. Vol. II deals principally with series and various ways of representing functions. The theory of trigonometric series, so important in the modern development of the theory of real functions and the beginning of the author's interest in the general field of real functions, constitutes the longest chapter, namely VIII, which covers nearly 250 pages and includes some of the latest developments accessible to the author at the time of writing. Among the numerous topics discussed in Vol. II are, for example, the summation methods of Cesàro,

Hölder, and Riesz, Baire's results on the representation and classification of functions—the considerations here are based on the work of de la Vallée-Poussin—the integration theories of Tonelli and Perron, the construction of functions with assigned singularities, and normal orthogonal functions, in particular the extension to the latter of the Parseval and Riesz-Fischer theorems. Chapter VII, which deals with the representation of functions as limits of integrals, gives special prominence to one of the author's own results, the general convergence theorem, which asserts, under certain conditions, the convergence to zero, as $n \to \infty$, of an integral of type $\int_a^b f(t)\phi(t, x, n,)dt$, and has application to, as it is a summary and extension of, certain known facts concerning integrals occurring, for example, in the theory of Fourier's series.

It is somewhat regrettable that a work devoted to the theory of real functions in its broadest aspects should say so little of properties of unconditioned functions, especially when properties of particular functions, for instance the continuous, may repeatedly be exhibited as specialized cases of properties of arbitrary functions.

Whatever personal impressions one may have of this grand work of Hobson, one must be grateful for the painstaking scholarship of the author, who has made readily accessible to mathematicians the conceptual structure of the theory of real functions and the rich store of its most recent developments.

HENRY BLUMBERG

The New Quantum Mechanics. By George Birtwistle. Cambridge University Press, 1928, xiii+290 p.

The author,[1] a fellow of Pembroke College, Cambridge, England, has already written a book on the Bohr theory of quanta,[2] and now offers an exposition of the quantum theory developed since 1925, known as quantum mechanics. This account is very accurate and contains practically everything that has been done up to the summer of 1927. He gives us, so to speak, original abstracts of the principal papers and allows us a survey of everything that is known. This makes the work not an exposition from one point of view, as is Weyl's new book; it is rather an "impartial" treatment of the methods of the different schools, with credit given to each for its results.

From Birtwistle's book we see quantum mechanics developing in the following fashion. Some years before 1925 Sommerfeld and others had been grappling with the complex problem of the multiplets in spectra and their Zeeman separations. It seemed impossible to account for them on the basis of the Bohr theory of quanta. This theory, unsatisfactory from the beginning through its theoretical inconsistencies, showed itself further insufficient in practical respects.

This led the young German physicist, Heisenberg, to an investigation in

[1] The author died on May 19, 1929 at the age of 52.

[2] *The Quantum Theory of the Atom*, Cambridge, 1926.

1925, in which he developed, from other principles than those of Bohr, a new quantum mechanics, discarding Newtonian mechanics. Heisenberg's leading principle was that the quantum formulae had to contain only experimentally observable magnitudes, such as the frequencies and the intensities of the spectrum. He did not succed in all respects, but his theory was much better than the previous one with its radii of orbits, orbital frequencies, and orbital amplitudes, which probably by their nature can never be observed.

The mathematical tool of the new theory was the so-called matrix calculus, later developed by Heisenberg, himself, Born, and Jordan. In the same period the English physicist Dirac worked out an analogous theory, the so-called q-numbers. The crucial test of both theories was the computation of the lines of the hydrogen spectrum, in which both succeeded.

It was now possible to attack the difficulties in the multiplets. In 1926 Heisenberg and Jordan succeeded in clearing them up on the basis of the new mechanics. They used in their investigation a new notion: of the so-called spinning electron, introduced by Goudsmit and Uhlenbeck (now at Ann Arbor), according to which an electron can spin around like a top.

The new matrix theory was very complicated, its physical meaning very obscure. But in 1926 a quite new development was put forward by Schrödinger, at that time professor at Zürich. Inspired by the ideas of Louis de Broglie's Paris thesis of 1924, in which a wave was associated with every material particle, Schrödinger assumed that the dynamics of an electron can not be those of a point as in classical theory, but must be those of a wave. This wave obeys a linear partial differential equation of the second order. (Newton mechanics leads to a partial differential equation of the first order and second degree.) Such a differential equation admits continuous uniform bounded solutions only for certain discrete values of a parameter (the "energy"), the so-called characteristic values. The critical test of the hydrogen lines was also satisfied in Schrödinger's case.

It was soon clear that the matrix theory of differential equations led to the same results, though the starting point was entirely different. Schrödinger soon showed the reason for this. The main difficulty remained the interpretation of both theories. Schrödinger's way had the advantage of using the highly develloped analysis of linear partial differential equations, and therefore had mathematical advantages. Heisenberg used Schrödinger's calculus to solve his matrix equations for the case of the helium atom, where there are two electrons instead of the one electron in the hydrogen atom.

The reason that Schrödinger's calculus was not victorious was the difficulty in the interpretation. He could show that in highly excited states a suitably chosen group of characteristic functions represent a "wave packet" behaving like a point mass of classical mechanics oscillating in a rectilinear path. Heisenberg showed, however, that in general such a wave packet is not stable, that it spreads out over the whole space in the course of time. Schrödinger's theory cannot, therefore, account for mass points.

The matrix school, on the contrary, preferred a statistical interpretaion of the results. What Schrödinger and his school called the "density," is for Heisenberg and his school a "probability that an electron is in a certain state." The laws of quantum mechanics become, in this theory, statistical laws, but laws of another mathematical structure than the laws of the ordinary theory of probability, underlying for instance the kinetic theory of gases.

Statistics become, therefore, an, essential part of the new mechanics. Here exist three hypotheses, already, on the basis of the old quantum theory, and its application to the theory of radiation; the theories named after Einstein and Bose, Boltzman, and Fermi and Dirac. The second underlies the kinetic theory of gases of Newtonian mechanics. The interpretation of these hypotheses in the many electron problems of the new quantum mechanics is only in a very elementary stage of development.

This is the main content of Birtwistle's book. He ends with an exposition of Heisenberg's "indetermination principle" (1927). One of the characteristics of the new theory is, according to this principle, that it is essentially impossible to get an exact statement about the momentum of an electron if we want an exact statement about its position, and vice versa; the same holds for time and energy, and for all variables and their Fourier transforms. This is a simple corollary of the duality between point-conception and wave-conception.

In a theory in such a rapid state of development a book like Birtwistle's is already incomplete after a year has passed. We will mention here some of the outstanding developments published later than the book: Weyl's investigations on finite groups and quantum mechanics, collected in his new book, *Quantenmechanik und Gruppentheorie*; Dirac's relativistic theory of the spinning electron; Eddington's connection between the so-called exclusion principle of Pauli and the Fermi-Dirac statistics; and the development of the many-electron problem.

Birtwistle's book does not mention the experimental results showing the wave-character of electrons, as do those of Davisson, Germer, Thomson, and others.

D. J. STRUIK

General Mathematics. By C. H. Currier and E. E. Watson. The Macmillan Company, New York, 1929. viii+413 pages.

The first brief review of the book appears, where it properly should, in the authors' preface. We quote from it without their permission: "This book includes the elements of algebra, trigonometry, analytic geometry, and calculus · · · · The review topics, such as the graph, simultaneous equations · · · · are interwoven with and related to the conic sections and other more advanced material. The chief features of the book are the simplicity of method and the selection and order of presentation of the subject matter. The problem material has been selected with reference to utility and interest rather than mathematical completeness. Methods and problems which have proved unsatisfactory [as

tried out in the authors' classrooms] have been eliminated. It is therefore the hope of the authors that this book will prove to be teachable in the hands of others." ....

As textbooks begin to flood the market, reviewers, more especially those who have "authored" one themselves, question the need or the whys for another textbook in the field. "What is there about the newcomer that justifies its crowding the others in the field?" is the usual query. It is many years now since that question has ceased to be original. In the minds of most of us a new one has arisen: "How long will it be before each of us has written his own textbook?" And why not? Almost ever new textbook appears only after it has been "tried out successfully in the author's own classes." Almost every new author claims, usually sincerely and justifiably, that not a textbook on the market meets the needs of either his special style of teaching or the unique requirements of his students. Unwilling to sacrifice himself or his students, what is there left to do but write his own?

Hence, in all seriousness, the reviewer feels that *General Mathematics* by Currier and Watson meets a worth-while need. After a careful reading of the book he finds nothing strikingly objectionable either in content or method. On the other hand it is not likely to set the freshman mathematics world aflame. (Nor do the authors make any such claim.) The reviewer hopes,with the authors, that others may find the book teachable, for despite the ever increasing number of textbooks the teachable ones are scarce indeed.

Evidently the authors' intention is to treat the various branches of elementary mathematics simultaneously. Theirs is distinctly *not* a textbook in "unified", "correlated," or "joined" mathematics. Chapters in algebra are interspersed with those on trigonometry, analytic geometry, calculus, and statistics. The following are the chapter headings given in the order of their appearance:

Functions and Graphs; Trigonometric Functions of an Acute Angle; Exponents—Logarithms; Radian Measure—Trigonometric Functions of Any Angle; Straight Line Formulas; The Quadratic Function; Theory of Equations; First Degree Equations—Use of Determinants; Differentiation of Algebraic Functions; Integration; Relations Among Trigonometric Functions; Polar Coordinates and Allied Topics; Progressions and Series—Interest Formulas —Binomial Theorem; Laws of Growth—Exponential Function; Conic Sections; Space of Three Dimensions; Permutations and Combinations; Theory of Measurements; Complex Numbers.

In all, approximately 130 pages are devoted to algebra, 65 pages to trigonometry, 100 pages to analytic geometry, 65 pages to calculus, and 25 pages to statistics.

The exercises on the whole, are quite excellent. A representative number are "practical" without being too technical. Historical notes abound and in most cases are appropriate and stimulating.

The presentation of new material is all too frequently sketchy and too many proofs are elliptical, to meet the needs of the average student. In partic-

ular, the authors fail to smooth the two rough spots in introductory analytic geometry: (1) Expressing geometric magnitudes or configurations in terms of general or particular coordinates of significant points; (2) "answers" in which x and y remain to the bitter end, thus involving a new notion directly opposed to that of elementary algebra.

Occasionally the lettering of diagrams is not uniform or is too involved. Consider, for example, figure 66, page 95: since a brace is used to indicate the whole of the line y_2-y_1, why is not as much done for x_2-x_1? Also, if the diagram is to aid or elucidate the proof, why not let the student *see* that AP is $y-y_1$, that CP_2 is y_2-y_1, etc.?

What is gained in algebra or analytic geometry by the use of the more or less technical words "rise" and "run" (p. 86)? In $A=\pi r^2$, or $C=2\pi r$, r is *not* an arbitrary constant (page 1). The example on page 12 is confusing. Perhaps the fifth line should read: "If the center of *a* circle is the origin $\cdots$" On p. 41 the term "significant figures" is used without definition or explanation. And lastly, it seems a bit unusual to have a chapter on "Complex Numbers" come last (pages 375–386). Pages 387–392 contain six brief tables. Answers to practically all the exercises are given on pages 397–413.

JOSEPH SEIDLIN

Operational Circuit Analysis. By V. Bush. John Wiley and Sons, New York, 1929. x+392 pages.

In the introduction to this book the author wishes it to be emphasized that he writes as an engineer and that he does not pretend to be a mathematician. However he is writing about a mathematical subject, the solution of linear differential equations (mainly with constant coefficients), and this review is to appear in a mathematical journal, so that we are constrained to regard the mathematical aspects of the book. These we are sorry to say do not command our admiration.

The material treated is Heaviside's operational method together with the more modern modifications, of which the most important is Carson's "infinite integral." The material in pages 1–147 and 189–263 of the book is in our opinion treated more thoroughly and scientifically in an eighteen page article by Paul Lévy.[1] The rest of the book is devoted to treatment by methods of complex variable theory and Fourier integrals. We are inclined to the opinion, purely personal, that the complex variable sortie, led by Wagner, Bromwich, et al., was ill advised. Heaviside's own ideas were more direct and the perfectly rigorous explanation of these (based on Volterra's concept of composition) given by Lévy leaves nothing to be desired. Particularly in the matter of asymptotic solutions there is an irritating feeling through Bush's book of "now you see it, now you don't" (pp. 242–255). The chapter on networks with variable parameters deals with very interesting questions. It is not at all clear to

[1] *Le calcul symbolique d'Heaviside*, Bulletin des sciences mathématiques (2) vol. **50** (1926).

us that the function A of equation (714) will be a function of the difference $t-\lambda$; when the coefficients of the differential equations involved depend on the time t this function A, whose determination solves, in effect, the problem will be a function of both t and λ and not merely of their difference. It is this latter characteristic of equations with constant coefficients that makes them so easy to deal with. In the case in question the variable coefficient arises from a variable resistance and the artifice of imagining the current short circuited across it is resorted to; the result may be correct but it is certainly *not proved.*

Now that we have given our opinion of the book as a mathematician we may be permitted to add that the engineering information given in the book is very good particularly for mathematicians who are unfamiliar with these applications. The book is conscientiously written, well printed, and we feel sure that in engineering circles it will secure a good reputation.

F. D. MURNAGHAN

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEM FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3404. *Proposed by W. J. Greenstreet, Editor of the Mathematical Gazette.*

An endless inextensible weightless string lies in the form of a rhombus $ABCD$; masses each of m lbs. weight are placed at the corners. The particle at A is struck along the diagonal CA away from A by a blow P. What will be the initial velocity of the particle at C?

3405. *Proposed by Paul Wernicke, Washington, D. C.*

Given in a plane three concurrent lines and a point P. Construct an equilateral triangle having its vertices on the three lines and having the point P on one of its sides. Determine the number of solutions.

3406. *Proposed by William P. Parker, Pyengyang, Chosen, Korea.*

Find the condition that $(\alpha x+\beta y+\gamma z)\cdot(\alpha_1 x+\beta_1 y+\gamma_1 z)-(\alpha_2 x+\beta_2 y+\gamma_2 z)^2$ may be resolved into two linear factors.

3407. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

A ray of light emanating from a fixed source L is reflected by a flat mirror

at a point M so that the reflected ray passes through a given point I. Find the locus of M when the mirror revolves about a fixed axis s.

3408. *Proposed by J. V. Uspensky.*

Show that the integral

$$V_n = \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n$$

converges to the limit $2/3$ when n increases indefinitely and that the product $n(V_n - 2/3)$ remains bounded.

SOLUTIONS

263 [1917, 177]. *Proposed by J. L. Riley.*

To find positive integral values which verify the equation $x^3+2=y^2$.

Note by the Editors: A solution of this problem is given in the paper, "A Note on the Solution of $x^3+2=y^2$," by P. H. Daus in the Bulletin of the American Mathematical Society, vol. 35 (1929), p. 597. See also the solution in this Monthly by A. Brauer [1928, 494], and the partial solution by L. Hampton [1928, 322].

3333 [1928, 377]. *Proposed by W. H. Roever, Washington University.*

Consider the system of two partial differential equations

$$X_i(f) \equiv a_{1i}\frac{\partial f}{\partial x_1} + a_{2i}\frac{\partial f}{\partial x_2} + a_{3i}\frac{\partial f}{\partial x_3} = 0, \quad i = 1,2,$$

in which the coefficients a_{ji} are functions of x_1, x_2, x_3. From one point of view, a necessary and sufficient condition that these equations have a common integral surface is

$$\Theta \equiv \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ X_1(a_{12}) - X_2(a_{11}) & X_1(a_{22}) - X_2(a_{21}) & X_1(a_{32}) - X_2(a_{31}) \end{vmatrix} \equiv 0,$$

and from another point of view, such a condition is

$$\Omega \equiv U\left(\frac{\partial V}{\partial x_3} - \frac{\partial W}{\partial x_2}\right) + V\left(\frac{\partial W}{\partial x_1} - \frac{\partial U}{\partial x_3}\right) + W\left(\frac{\partial U}{\partial x_2} - \frac{\partial V}{\partial x_1}\right) \equiv 0,$$

in which

$$U = \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}, \quad V = \begin{vmatrix} a_{31} & a_{11} \\ a_{32} & a_{12} \end{vmatrix}, \quad W = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}.$$

Show that $\Theta \equiv \Omega$.

Solution by Otto Dunkel, Washington University.

In the determinant Θ multiply the elements of the first row by $-[\partial a_{12}/\partial x_1$

$+\partial a_{22}/\partial x_2+\partial a_{32}/\partial x_3]$, and those of the second row by $[\partial a_{11}/\partial x_1+\partial a_{21}/\partial x_2 +\partial a_{31}/\partial x_3]$; add the corresponding products to the elements of the third row. After certain cancellations the resulting third row becomes

$$\partial V/\partial x_3 - \partial W/\partial x_2,\ \partial W/\partial x_1 - \partial U/\partial x_3,\ \partial U/\partial x_2 - \partial V/\partial x_1,$$

and the determinant is now in the form Ω.

3349 [1928, 494]. *Proposed by S. A. Corey, Des Moines, Iowa.*

If $|lmn|$ denote the determinant whose columns are the lth, mth, and nth columns of the array

$$\begin{vmatrix} y & x & -av & au \\ u & bv & x & -by \\ v & -cu & cy & x \end{vmatrix},$$

prove that

$$(x^2 + bcy^2 + acu^2 + abv^2)^3 = |234|^2 + bc\,|134|^2 + ac\,|124|^2 + ab\,|123|^2.$$

Solution by the Proposer.

On expanding the given determinants $(x^2+bcy^2+acu^2+abv^2)$ is found to be their common factor. If we denote this factor by D, the given equation becomes $D^3=x^2D^2+bcy^2D^2+acu^2D^2+abv^2D^2$, an identity.

3357 [1928, 564]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Through a given point to draw a line so that the segment intercepted on it by two given intersecting lines shall be divided in a given ratio by the foot of the perpendicular dropped upon it from the point of intersection of the two given lines.

Solution by Harry Pool, South Dakota State School of Mines.

Suppose the two given intersecting lines are l and m, meeting in a point O. Let the given point, through which the required line is to be drawn, be P.

First construct a triangle whose vertical angle is equal to the angle at O, and whose base is divided in the given ratio by the perpendicular upon it from the vertex. To do this, on any line AB describe a circular segment containing the angle O. Divide the line AB in the given ratio at D. Draw DC at right angles to AB to meet the arc of the segment at C. Then ABC is the triangle required.

Through any point on l draw a line n making an angle equal to angle CAB of the triangle just constructed. The line through P parallel to line n is the line required by the problem.

Also solved by L. W. Johnson, Mark Kurtz, Mark Landan, J. Q. McNatt, W. V. Parker, and S. Pelletier.

3359 [1928, 564]. *Proposed by Clifford N. Mills, Normal, Illinois.*

A, B, C, D are four points in a plane. Let the centroids of the triangles BCD, CDA, DBA, ABC be respectively a, b, c, d. Then Aa, Bb, Cc, Dd meet in

a point, and are divided in the same ratio at this point. Also the quadrilateral *abcd* is similar to *ABCD*.

I. *Solution by T. H. Butchart, Urbana, Ill.*

Consider the triangle ABM, where M is the mid-point of DC. Then b and a lie on AM and BM, respectively, so that $Mb/MA = Ma/MB = ab/AB = 1/3$. Similar reasoning applied to bc, cd, da, shows that $abcd$ has its sides parallel and equal to one third of the corresponding sides of $ABCD$. The two quadrilaterals are therefore similar and similarly placed, and hence the lines joining their corresponding vertices meet in a point O such that

$$Oa/OA = Ob/OB = Oc/OC = Od/OD = 1/3.$$

II. *Solution by E. C. Kiefer, Decatur, Ill.*

Placing the quadrilateral with the side AD on the x-axis and the point A at the origin, let the coördinates of the points be $B(x_1, y_1)$, $C(x_2, y_2)$, $D(x_3, 0)$. Then the equations of the lines are

$$\begin{aligned}
Aa,&\quad x(y_1 + y_2) - y(x_1 + x_2 + x_3) = 0,\\
Bb,&\quad x(y_2 - 3y_1) - y(x_2 + x_3 - 3x_1) + y_1(x_2 + x_3) - x_1y_2 = 0,\\
Cc,&\quad x(y_1 - 3y_2) - y(x_1 + x_3 - 3x_2) + y_2(x_1 + x_3) - x_2y_1 = 0,\\
Dd,&\quad x(y_1 + y_2) - y(x_1 + x_2 - 3x_3) - x_3(y_1 + y_2) = 0.
\end{aligned}$$

Since the sum of the left sides of these equations is identically zero, the four lines meet in the same point; the first and fourth equations give for its coördinates $\frac{1}{4}(x_1+x_2+x_3)$, $\frac{1}{4}(y_1+y_2)$. This point is easily shown to divide each of the segments Aa, Bb, Cc, Dd in the ratio of 3:1; it is therefore a center of similitude for the figures $ABCD$ and $abcd$.

Also solved by Rufus Crane, P. S. Dwyer, H. A. DoBell, S. E. Field, A. E. Gault, J. H. Neelley and A. Pelletier.

3361 [1929, 105]. *Proposed by B. F. Finkel, Drury College.*

Find the envelope of a system of circles having for diameter a secant of constant length, $2r$, of a conic.

Solution by Otto J. Ramler, Catholic University, Washington, D. C.

Consider the case of an ellipse with the equation $b^2x^2+a^2y^2-a^2b^2=0$; and let the chord with the length $2r$ have the extremities (x_1, y_1), (x_2, y_2), where

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = 4r^2. \tag{1}$$

Then from the equation of the ellipse we have $b^2(x_2^2-x_1^2)+a^2(y_2^2-y_1^2)=0$. If m is the slope of the chord and (x, y) is its middle point, this equation may be written:

$$b^2x + ma^2y = 0. \tag{2}$$

From (1) and the equation of the chord we have

$$(3) \qquad (y_2 - y_1)/m = x_2 - x_1 = 2r(1 + m^2)^{-1/2}.$$

Combining these results with $2x = x_2 + x_1$, $2y = y_2 + y_1$, we find

$$(4) \qquad y_2 = y + mr(1 + m^2)^{-1/2}, \quad x_2 = x + r(1 + m^2)^{-1/2}.$$

Substituting these values of x_2, y_2 in the equation of the ellipse and eliminating m by means of (2), we obtain

$$(5) \qquad (b^2x^2 + a^2y^2 - a^2b^2)(b^4x^2 + a^4y^2 + a^2b^2r^2) + a^4b^4r^2 = 0,$$

the equation of the locus of the centers of the variable circles.

The envelope of the circles of radius r and centers (x, y) will consist of two curves parallel to (5) whose points are the extremities of the normal to (5) with the constant length r.

For the case of the hyperbola we have merely to replace b^2 in (5) by $-b^2$. The equation resulting from the consideration of a parabola may be obtained in the same manner.

Note by the Editors: This problem is the same as 3310 [1928, 154], a solution of which by W. J. Patterson has been printed [1929, 233].

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus Ohio.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Fourth Carus Monograph on Projective Geometry by Professor J. W. Young of Dartmouth College is now in page proof and will be issued in a few weeks. Members, individual and institutional, should order now through Secretary Cairns at the cost price of one dollar and twenty five cents per copy.

The Nobel Prize in Physics for 1928 has been awarded to Professor O. W. Richardson, of Kings College, London. The prize in physics for 1929 has been awarded to the Duc de Broglie, of Paris.

On the occasion of its 175th anniversary, Columbia University conferred the degree of honorary doctor of philosophy on Dean H. E. Hawkes and Professors C. J. Keyser and D. E. Smith, among others.

Doctor Max Mason has been elected president of the Rockefeller Foundation.

Professor Edwin B. Wilson, of Harvard University, has been elected president of the Social Science Research Council.

Dr. F. R. Bamforth has been appointed assistant professor of mathematics at Cornell University.

Professor M. A. Brumbaugh, of the University of Pennsylvania, has been appointed professor of statistics in the School of Business Administration of the University of Buffalo.

Assistant Professor D. L. Holl, of Iowa State College, has been promoted to an associate professorship.

Mr. W. H. McEwen, who has been teaching mathematics at Regina College Regina, Saskatchewan, is spending the year in graduate study at the University of Minnesota.

Mr. R. H. Marquis has been appointed assistant professor of mathematics at Ohio University, Athens, Ohio.

Dr. E. L. Mickelson has been appointed professor and head of the department of mathematics at New Mexico State Teachers College.

Professor H. W. Tyler, Secretary of the American Association of University Professors, having returned to the Massachusetts Institute of Technology on the expiration of his leave of absence, the management of the Washington office has been assumed by the treasurer, Professor Joseph Mayer, who has been granted leave of absence by Tufts College for this purpose.

A New Publishing House

RICHARD R. SMITH *Inc.*

has opened offices at 12 East 41st Street, New York City. Mr. Smith, who recently resigned as a Director of the Macmillan Company, has for the past twenty years been in charge of the College Department of Macmillan's and, according to The *Publishers' Weekly*, "built up a list of books recognized in both publishing and educational circles as of unquestioned leadership with a volume of business exceeding in amount the entire business of many publishers."

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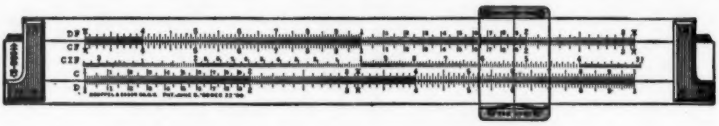

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DIRECTORY

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BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.
INDIANA, Earlham College, May 2-3.
IOWA.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Cleveland, Miss., April.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN.
MINNESOTA.
MISSOURI.
NEBRASKA, Peru, Neb., May 9.
OHIO, Columbus, Ohio, April 3.
PHILADELPHIA.
ROCKY MOUNTAIN.
SOUTHEASTERN.
SOUTHERN CALIFORNIA.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE MATHEMATICAL ASSOCIATION OF AMERICA

The following twenty-nine persons and one institution have been elected to membership in the Association on applications duly certified:

To Individual Membership

R. A. BEAVER, A.M. (Columbia). Instr., N. Y. State Coll., Albany, N. Y.

A. D. BRADLEY, A.M. (Teachers Coll., Columbia). Instr., Hunter Coll., New York, N. Y.

J. F. BUTLER, A.M. (St. Louis Univ.). Instr., Milford Novitiate, Milford, Ohio.

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To Institutional Membership

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THE THIRTEENTH ANNUAL MEETING OF THE MISSOURI SECTION

The thirteenth annual meeting of the Missouri Section of the Mathematical Association of America was held at Washington University, St. Louis, on Saturday morning, November 16, 1929. Professor Byron Ingold presided. The at-